The Dold-Kan Correspondance

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Abstract

We can construct a category of simplicial abelian groups in a similar way of simplicial sets, there is a natural free forgetful adjunction $U\colon \mathbf{sAb} \rightleftarrows \mathbf{sSet}\colon F$. We then see how we can naturally construct a non-negatively graded chain complex given a simplicial abelian group called the normalized complex. This definition is functorial and has an inverse up to natural isomorphism creating an adjoint equivalence of categories $N\colon \mathbf{sAb} \leftrightarrows \mathbf{Ch}_{\geqslant}\colon \Gamma$ These notes support a talk given at the end of a course in Algebraic Topology. We will assume knowledge of the fundamental group, homology triangulations, simplicial sets and basic category theory. A good reference for Algebraic Topology is Hatcher [2] and for basic category theory is Riehl [3]. The main reference for these notes is Goerss-Jardine [1].

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1 Simplicial Abelian Groups

First we recall some notions for defining simplicial sets.

Definition 1.1. Let Δ denote the *simplex category* with:

1. Objects are posets, $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ for all $n \in \mathbb{N}$.
2. Morphisms are all monotone maps, $\Delta([n],[m]) = \{f\colon [n] \to [m] | f(a) \leq f(b) \text{ for } a \leq b\}$
3. identities and composition is as for functions.

Remember that we defined simplical sets as the presheaf over this category, that is $\mathbf{sSet} = \mathbf{Set}^{\Delta^{op}}$. We can consider the $\mathbf{Ab}$-valued presheaf that is,

Definition 1.2. Let Δ be the simplex category as above, denote the $\mathbf{Ab}$-*valued presheaf over* Δ be $\mathbf{sAb} = \mathbf{Ab}^{\Delta^{op}}$.

Notice that objects of $\mathbf{sAb}$ are functors $A\colon \Delta^{op} \to \mathbf{Ab}$ and morphims are natural transformations. The data of a simplicial abelian group A can be understood as a collection of Abelian groups A_n and $n+1$ face and degeneracy group homomorphisms

$$\delta_i\colon A_n \to A_{n-1},$$

and

$$\sigma_i\colon A_{n-1} \to A_n,$$

respectively for all $n \in \mathbb{N}$. We will call a simplex $a \in A_n$ *degenerate* if $a = \sigma_i(\alpha)$ for some $\alpha \in A_{n-1}$ and a *face* if $a = \delta_i(\alpha)$ for some $\alpha \in A_{n+1}$.

Remark 1.3. Notice here that we could have done the same construction using $\mathbf{R-mod}$ or $\mathbf{Grp}$ to get categories $\mathbf{sR-mod}$ or $\mathbf{sGr}$ in which case we would get $\mathbf{s}\mathbb{Z}-\mathbf{mod} \cong \mathbf{sAb}$

We will state the following simplicial identities without proof, which are a consequence of the combinatorial definition of simplical sets.

Lemma 1.4. Suppose $A \in \mathbf{sAb}$ is a simplicial abelian group with face and degeneracy maps $\delta_i\colon A_n \to A_{n-1}$ and $\sigma_i\colon A_{n-1} \to A_n$. We have the following simplicial identities:

$$\begin{aligned}
\delta_i\delta_j &= \delta_{j-1}\delta_i && \text{for } i < j, \\
\delta_i\sigma_j &= \sigma_{j-1}\delta_i && \text{for } i < j, \\
&= 1 && \text{for } i = j \text{ or } i = j+1, \\
&= \sigma_j\delta_{i-1} && \text{for } i > j+1, \\
\sigma_i\sigma_j &= \sigma_{j+1}\sigma_i && \text{for } i \leq j.
\end{aligned}$$

Remark 1.5. We can use this lemma to show that every map in the simplex category $[m] \to [n]$ can be factored in to a epi-mono map,

$$[m] \xrightarrow{t} [r] \xhookrightarrow{d} [n] \ .$$

This makes sense intuitively by following all the co-face maps followed by the co-degeneracies.

Definition 1.6. There is a forgetful functor $U\colon \mathbf{sAb} \to \mathbf{sSet}$ which for each $n \in \mathbb{N}$ sends A_n to the underlying set of the group and each face and degeneracy group homomorphism to the underlying function. This functor has a left adjoint $F\colon \mathbf{sSet} \to \mathbf{sAb}$ which for each $n \in \mathbb{N}$ sends X_n to the free group on the set X_n.

2 The equivalence

Recall the definition a chain complex.

Definition 2.1. A *chain complex over an abelian category* $\mathscr{A}$, (C, ∂) is a collection of objects of $\mathscr{A}$ with morphisms $\partial_i\colon A_i \to A_{i-1}$ such that $\partial_{i-1}\partial_i = 0$. A *chain map* between chain complexes $f\colon C \to D$ is a morphism of $\mathscr{A}$ in each degree $f_n\colon C_n \to D_n$ such that $f\partial = f\partial$ (dropping the decorations for ∂). We have a category $\mathbf{Ch}_{\geqslant}$ which has as objects chain complexes, morphisms chain maps and the obvious composition and identity.

2.1 Normalised complex

We first notice the normalised complex which is the natural chain complex which can be defined as follows,

Definition 2.2. Let $A \in \mathbf{sAb}$ be a simplicial abelian group. The *normalised complex* NA is defined for each $n \in \mathbb{N}$,

$$NA_n = \bigcap_{i=0}^{n-1} ker(\delta_i) \subset A_n$$

where $\delta_i\colon A_n \to A_{n-1}$ are the face maps of A_n and $\partial_n = (-1)^n\delta_n\colon A_n \to A_{n-1}$ This construction defines a functor $N\colon \mathbf{sAb} \to \mathbf{Ch}_{\geqslant}$ where a natural transformation of simplicial abelian groups is mapped to the obvious mapping.

We have that $\partial_{n-1}\partial_n = (-1)^{n-1}\delta_{n-1}(-1)^n\delta_n = \delta_{n-1}\delta_{n-1} = 0$ via the simplicial identities above, hence (NA, ∂) is a well defined chain complex. Also check that $N(\eta \circ \varepsilon) = N(\eta) \circ N(\varepsilon)$ and $N(id_A) = id_{NA}$ and so this is truly functorial.

Remark 2.3. In this definition it is assumed that the abelian category we are working with is $\mathbf{Ab}$ of abelian groups. Hence $\mathbf{Ch}_{\geqslant}$ is the category of chain complexes over $\mathbf{Ab}$.

You may see that it is more natural to define the following chain complex called the Moore complex. We will see how these are related.

Definition 2.4. Let $A \in \mathbf{sAb}$ be a simplicial abelian group. Define the *Moore complex*, (A, ∂) for each $n \in \mathbf{N}$ as A_n with boundary,

$$\partial = \sum_{i=0}^{n} (-1)^i d_i\colon A_n \to A_{A-1}$$

Again $\partial^2 = 0$ is a consequence of the identities in Lemma 1.4. Also remark the slight abuse in notation. Let $DA_n \leq A_n$ be the subgroup generated by the degenerate simplicies in A_n. We can define a quotient complex A/DA which has natural inclusion and projections,

$$NA \xrightarrow{i} A \xrightarrow{p} A/DA$$

Which leads to the following proposition.

Proposition 2.5. The map $pi\colon NA \to A/DA$ is an isomorphism.

Proof. Let $N_jA_N = \bigcap_{i=0}^{j} ker(\delta_i) \subset A_n$. Proceed via induction on the map,

$$N_jA_n \xrightarrow{i} A_n \xrightarrow{p} A_n/D_jA_n$$

□

Suppose A is a simplical abelian group every simplical map $d^*\colon A_n \to A_m$ which comes from a simplex monomorphism $d\colon [m] \hookrightarrow [n]$ induces a map in the normalised complex $d^*\colon NA_n \to NA_m$. However, looking at how NA is defined we see $d^* = 0$ if $m \neq n-1$. This leads us to consider the following. Suppose we have a chain complex (C, ∂) then for each $d\colon [m] \to [n]$ we define

$$d^* = \begin{cases} (-1)^n\partial & \text{if } d\colon [n-1] \to [n] \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.6. Define a functor $\Gamma\colon \mathbf{Ch}_{\geqslant} \to \mathbf{sAb}$ on object (C, ∂) as:

$$C_n \mapsto \bigoplus_{s\colon [n] \twoheadrightarrow [k]} C_k$$

Where $s\colon [n] \twoheadrightarrow [k]$ is a surjective map in the simplex category. For this to be a simplicial abelian group we define for each map in the simplex category $\theta\colon [m] \to [n]$ we have a group homomoprhism, $\theta^*\colon \Gamma(C)_n \to \Gamma(C)_m$ defined by as

$$C_k \xrightarrow{d^*} C_l \xrightarrow{int} \bigoplus\nolimits_{[m]\twoheadrightarrow[r]} C_r$$

Where d^* is the map induced by the factorization of

$$[m] \xrightarrow{\theta} [n] \xrightarrow{s} [k]$$

into,

$$[m] \xrightarrow{t} [l] \xhookrightarrow{d} [k]$$

One checks this is a functor by stating the obvious morphims of maps and checking functorality conditions.

Theorem 2.7. $N\colon \mathbf{sAb} \to \mathbf{Ch}_{\geqslant}$ and $\Gamma\colon \mathbf{Ch}_{\geqslant} \to \mathbf{sAb}$ as defined above are inverse upto natural isomorphism. Hence $\mathbf{sAb}$ and $\mathbf{Ch}_{\geqslant}$ are equivalent as categories.

Proof. The full proof can be found in Goerss-Jardine [1], here we give an outline. Notice that,

$$D\Gamma(C)_n = \bigoplus_{s\colon [k]\twoheadrightarrow[n], k\leq n-1} C_k$$

And so we have a natural isomorphism,

$$C \cong M\Gamma(C)/D\Gamma(C) \cong N\Gamma(C).$$

The idea for the other isomorphism is we have a natural map

$$\Psi\colon \Gamma NA \to A$$
$$\bigoplus_{s\colon n\twoheadrightarrow k} NA_k \mapsto A_n$$

Where on each summand,

$$NA_k \hookrightarrow A_k \xrightarrow{\sigma} A_n$$

Where σ is the homomorphism induced by s. Notice $N(\Psi)$ is an isomorphism. Then show N is exact and preserves epimorphisms and Ψ is surjective in all degrees and it follows Ψ is an isomorphism. □

Remark 2.8. This equivalence holds for any abelian category $\mathscr{A}$.

References

[1] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Prog. Math.* Basel: Birkhäuser, 1999.

[2] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.

[3] Emily Riehl. *Category theory in context*. Mineola, NY: Dover Publications, 2016.