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Basic Category Theory

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Contents

1 Introduction

This is a short introduction on basic category theory with emphasis on proofs of results and examples to help illustrate the categorical perspective. We include definitions and examples of categories, functors and natural transformations we then take a deeper look into the idea of universal properties such as adjoints, limits, initial and terminal objects, and monads. We look at how these universal properties are linked as well as how they show up in categories we are used to working with.

Category theory is a branch of maths developed in the 1940's by Saunders Maclane and Samuel Eilenberg as an offshoot of algebraic topology ([11]). Category theory aims to generalise the construction of mathematical objects such as: mappings, products, quotient spaces algebras and modules.

Category theory is a vast area of mathematics and this report only states the very beginnings, interesting areas to look at after reading this Include: Representables and The Yoneda Lemma, Higher category theory and enriched category theory. This report has a ground up approach where we aim to prove every result stated. We also include lots of worked examples. The main references for this report are from the books: Adámek - Herrlick - Strecker [1], Leinster [5] and Riehl [9] some supporting theory is also from nCatLab [8] other references used will be cited in the report.

1.1 Ethics

It is important to uphold the academic integrity and ethical principals when writing a paper to ensure its credibility. In this paper we have, to our best ability, cited every author of books, webpages and other sources where we have adapted or used their ideas and material. We state where we have added our own proofs of theorems or examples. Additionally, we expect the publication of this report will cause no harm to human kind, animals or nature. The purpose of the paper is to compile ideas and research and add our own proofs and perspective on the examples to promote the advancement of knowledge in the field.

2 Categories

This section of notes adapted mainly from [5].

Definition 2.1. A *category* is a quadruple

$$\mathscr{C} = (\mathrm{Ob}(\mathscr{C}), \hom_{\mathscr{C}}(-,-), \circ_{(-,-,-)}, id_{-})$$

where:

1. $\mathrm{Ob}(\mathscr{C})$ is a *class*. We call the elements of $\mathrm{Ob}(\mathscr{C})$ *$\mathscr{C}$-Objects*;
2. $\hom_{\mathscr{C}}(-,-)$ is a *function*,
$$\begin{aligned}\hom_{\mathscr{C}}(-,-)\colon \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C}) &\to \mathbf{set},\\ (A,B) &\mapsto \hom_{\mathscr{C}}(A,B)\end{aligned}$$
where **set** is the class of all sets. The elements of $\hom_{\mathscr{C}}(A,B)$ we call maps from A to B;
3. Given $A,B,C \in \mathrm{Ob}(\mathscr{C})$, $\circ_{(A,B,C)}$ is a function,
$$\begin{aligned}\circ_{(A,B,C)}\colon\ \hom_{\mathscr{C}}(B,C) \times \hom_{\mathscr{C}}(A,B) &\to \hom_{\mathscr{C}}(A,C),\\ (g,f) &\mapsto g \circ_{(A,B,C)} f.\end{aligned}$$
We call each $\circ_{(A,B,C)}$ *composition*;
4. For each $A \in \mathrm{Ob}(\mathscr{C})$, id_A is an element of $\hom_{\mathscr{C}}(A,A)$.

The following axioms are satisfied:

1. $\circ$ satisfies *associativity*: Given $A,B,C,D \in \mathrm{Ob}(\mathscr{C})$, for each $f \in \hom_{\mathscr{C}}(A,B)$, $g \in \hom_{\mathscr{C}}(B,C)$ and $h \in \hom_{\mathscr{C}}(C,D)$ we have,
$$(h \circ_{(B,C,D)} g) \circ_{(A,B,D)} f = h \circ_{(A,C,D)} (g \circ_{(A,B,C)} f);$$
2. For each $A \in \mathrm{Ob}(\mathscr{C})$, the morphism id_A acts as an identity with respect to $\circ$. That is, given any two $\mathscr{C}$-Objects, $B,C \in \mathrm{Ob}(\mathscr{C})$ for all $f \in \hom_{\mathscr{C}}(A,B)$ and $g \in \hom_{\mathscr{C}}(C,A)$ we have:
$$f \circ_{(A,B,B)} id_A = f$$
and,
$$id_A \circ_{(B,A,A)} g = g.$$

Remark 2.2. The elements of each set $\hom_{\mathscr{C}}(A,B)$ where $A,B \in \mathrm{Ob}(\mathscr{C})$ are also called functions, arrows or morphisms, from A to B.

For simplicity we write $\circ$ for any $\circ_{(A,B,C)}$. We will also write $\hom$ if the category we are working with is clear.

Remark 2.3. In Definition 2.1 we have that each $\hom_{\mathscr{C}}(A,B)$ is a set, this definition can be changed so that these are not sets but classes, however in this report we will only use sets. In the literature categories defined as in Definition 2.1 may be refered to as *locally small categories*, see nCatLab [8] for more details.

2.1 Examples

We can now see how some of the structures we already know fit into the framework of a category.

Example 2.4. Let $\mathbf{Set} = (\mathrm{Ob}(\mathbf{Set}), \hom, \circ, id)$, where:

1. $\mathrm{Ob}(\mathbf{Set})$ is the class of all sets;
2. Given any two **Set**-Objects, (X, Y), $\hom(X, Y)$ is the set of all functions between X and Y;
3. Given three sets X, Y, Z,

$$\circ\colon \hom(Y, Z) \times \hom(X, Y) \to \hom(X, Z),$$
$$(f, g) \mapsto g \centerdot f$$

where $\centerdot$ is regular function composition;

4. For each $X \in \mathrm{Ob}(\mathbf{Set})$,

$$id_X\colon X \to X,$$
$$x \mapsto x.$$

Then $\circ$ is associative since function composition is associative and given any $X, Y \in \mathrm{Ob}(\mathbf{Set})$ then for all $f \in \hom(X, Y,)$ and $h \in \hom(Y, X)$:

$$f \circ id_X = f$$

and,

$$id_X \circ h = h.$$

Therefore **Set** is a category.

Example 2.5. Let **Grp** be defined:

1. $\mathrm{Ob}(\mathbf{Grp})$ is the class of all groups;
2. Given any two $(G, \bullet), (H, *) \in \mathrm{Ob}(\mathbf{Grp})$, $\hom((G, \bullet), (H, *))$ is the set of all group homomorphisms between $(G, \bullet)$ and $(H, *)$;
3. Given three groups $(G, \bullet), (H, *), (N, \square)$,

$$\circ\colon \hom((G, \bullet), (H, *)) \times \hom((H, *), (N, \square)) \to \hom((G, \bullet), (N, \square)),$$
$$(f, g) \mapsto g \centerdot f$$

where $\centerdot$ is regular function composition. This is well defined meaning $g \centerdot f$ is a group homomorphism;

4. For each $(G, \bullet) \in \mathrm{Ob}(\mathbf{Grp})$, $id_{(G,\bullet)}$ is the group homomorphism

$$id_{(G,\bullet)}\colon G \to G,$$
$$g \mapsto g.$$

Then $\circ$ is associative since function composition is associative and given any $\mathbf{Grp}$-Objects $(G, \bullet), (H, *) \in \mathrm{Ob}(\mathbf{Grp})$ then for all $f \in \hom((G, \bullet), (H, *))$ and $h \in \hom((H, *), (G, \bullet))$;

$$f \circ id_{(G,\bullet)} = f$$

and,

$$id_{(G,\bullet)} \circ h = h.$$

Therefore **Grp** is a category.

Example 2.6. Let K be a field then define $\mathbf{Vect}_K$ as:

1. $\mathrm{Ob}(\mathbf{Vect}_K)$ is the class of all K-vector spaces where K is a field;
2. For any two vector spaces V and W, $\hom(V, W)$ is the set of all K-linear maps between V and W;
3. $\circ$ is regular function composition;
4. For each $V \in \mathrm{Ob}(\mathbf{Vect}_K)$, id_V is the K-linear map:

$$\begin{aligned} id_V \colon V &\to V, \\ v &\mapsto v. \end{aligned}$$

Then $\mathbf{Vect}_K$ is a category since $\circ$ is associative and for any $f \in \hom(V, W)$ and $g \in \hom(W, V)$ $f \circ id_V = f$ and $id_V \circ g = g$.

The following example came from a lecture given by Dr Daniel Graves at the University of Leeds.

Example 2.7. Let $\mathbf{Top}^* = (\mathrm{Ob}(\mathbf{Top}^*), \hom, \circ, id)$ where:

1. $\mathrm{Ob}(\mathbf{Top}^*)$ Is the class of all based topological spaces, $((X, \tau_X), x_0)$;
2. Given two based topological spaces, $((X, \tau_X), x_0)$ and $((Y, \tau_Y), y_0)$, each

$$\hom_{\mathbf{Top}^*}(((X, \tau_X), x_0), ((Y, \tau_Y), y_0))$$

   is the set of continuous maps which sends $x_0 \mapsto y_0$ between these spaces;
3. $\circ$ is regular function composition. Given two base point preserving continuous maps, f, g, then $f \circ g$ is a base point preserving continuous map;
4. Given a based topological space $((X, \tau_X), x_0)$, id_X is the identity continuous map which sends each point $x \in X$ to itself.

$\mathbf{Top}^*$ is a category since $\circ$ is associative and each id_X acts as an identity.

2.1.1 Example: Category of monoids

Definition 2.8. A *monoid* is a triple $M = (M_s, *, e_M)$ where:

1. M_s is a set, we call the *underlying set of M*;

2. For any two elements $a, b \in M_s$, $*$ is a closed binary operation:

$$\begin{aligned} *\colon M_s \times M_s &\to M_s, \\ (a,b) &\mapsto a * b; \end{aligned}$$

3. e_M is an element in M_s such that for all $a \in M_s$,,

$$e_M * a = a * e_M = a;$$

4. $*$ is associative; Given any two $a, b, c \in M_s$

$$(a * b) * c = a * (b * c).$$

Remark 2.9. We will write $a \in M$ to mean $a \in M_s$.

Definition 2.10. Let $M = (M_s, *_M, e_M)$ and $N = (N_s, *_N, e_N)$ be monoids. A *monoid homomorphism* between M and N is a function, $f : M \to N$, which satisfies the following axioms:

1. For any $m_1, m_2 \in M$, $f(m_1 *_N m_2) = f(m_1) *_N f(m_2)$;
2. $f(e_M) = e_M$.

With monoids and their morphisms defined, we can define the category of monoids.

Lemma 2.11. Let **Mon** be defined:

1. $\text{Ob}(\textbf{Mon})$ is the class of all monoids as defined in Definition 2.8;
2. For each $M, N \in \text{Ob}(\textbf{Mon})$, $\hom(M,N)$ is the set of all monoid homomorphisms between M and N;
3. Given $M, N, H \in \text{Ob}(\textbf{Mon})$ $\circ$ is the function:

$$\begin{aligned} \circ\colon \hom(M,N) \times \hom(N,H) &\to \hom(M,H), \\ (f,g) &\mapsto g \centerdot f, \end{aligned}$$

where $\centerdot$ is regular function composition.

This is well defined since given any two monoid homomorphisms $f \in \hom(M,N)$ and $g \in \hom(M,H)$ and for each $x, y \in M$ we have,

$$\begin{aligned} g \centerdot f(x *_M y) &= g(f(x *^M y)) \\ &= g(f(x) *^N f(y)) \\ &= g(f(x)) *^H g(f(y)) \\ &= g \centerdot f(x) *^H g \centerdot f(y) \end{aligned}$$

and,

$$\begin{aligned} g \centerdot f(e_M) &= g(f(e_M)) \\ &= g(e_N) \\ &= e_H. \end{aligned}$$

Since f and g are monoid homomorphisms. Hence $g \centerdot h$ is a monoid homomorphism;

4. For each $M \in \mathrm{Ob}(\mathbf{Mon})$;

$$\begin{aligned} id_M \colon M_s &\to M_s, \\ m &\mapsto m. \end{aligned}$$

Then **Mon** is a category.

Proof. Firstly $\circ$ is associative since it defined as function composition.

To show the condition on identities,given any $f \in \hom(M, N)$ and $g \in \hom(N, M)$ we have for each $m \in M$ and $n \in N$;

$$\begin{aligned} f \circ id_M(m) &= f(id_M(m)) \\ &= f(m) \end{aligned}$$

and,

$$\begin{aligned} id_M \circ g(n) &= id_M(g(n)) \\ &= g(n). \end{aligned}$$

Hence id_M acts as an identity with respect to function composition and **Mon** is a category. □

The following Example 2.12 is adapted from [18].

Example 2.12 (Product Category)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories.

We can define the *Product category* $\mathscr{C} \times \mathscr{D}$ as follows:

1. The $\mathscr{C} \times \mathscr{D} - Objects$ are pairs (C, D) where $C \in \mathrm{Ob}(\mathscr{C})$ and $D \in \mathrm{Ob}(\mathscr{C})$;

2. For each $(C_1, D_1), (C_2, D_2) \in \mathrm{Ob}(\mathscr{C} \times \mathscr{D})$ the elements of $\hom_{\mathscr{C}\times\mathscr{D}}((C_1, D_1), (C_2, D_2))$ are the pairs (f, g) where $f \in \hom_{\mathscr{C}}(C_1, C_2)$ and $g \in \hom_{\mathscr{D}}(D_1, D_2)$;

3. Composition is defined,

$$\begin{aligned} \circ_{\mathscr{C}\times\mathscr{D}} \colon \hom_{\mathscr{C}\times\mathscr{D}}((C_2, D_2), (C_3, D_3)) &\to \hom_{\mathscr{C}\times\mathscr{D}}((C_1, D_1), (C_2, D_2)), \\ ((f_2, g_2), (f_1, g_1)) &\mapsto (f_2 \circ_{\mathscr{C}} f_1, g_2 \circ_{\mathscr{D}} g_1); \end{aligned}$$

4. Identities are defined for each $(C, D) \in \mathrm{Ob}(\mathscr{C} \times \mathscr{D})$:

$$id_{(C,D)} = (id_C, id_D).$$

First we show composition is associative: Given $(f_1, g_1) \in \hom_{\mathscr{C}\times\mathscr{D}}((C_1, D_1), (C_2, D_2))$, $(f_2, g_2) \in \hom_{\mathscr{C}\times\mathscr{D}}((C_2, D_2), (C_3, D_3))$ and $(f_3, g_3) \in \hom_{\mathscr{C}\times\mathscr{D}}((C_3, D_3), (C_4, D_4))$ we have,

$$\begin{aligned} ((f_3, g_3) \circ_{\mathscr{C}\times\mathscr{D}} \mathscr{C} \times \mathscr{D}(f_2, g_2)) \circ_{\mathscr{C}\times\mathscr{D}} (f_1, g_1) &= (f_3 \circ_{\mathscr{C}} f_2, g_3 \circ_{\mathscr{D}} g_2) \circ_{\mathscr{C}\times\mathscr{D}} (f_1, g_1), \\ &= (f_3 \circ_{\mathscr{C}} f_2 \circ_{\mathscr{C}} f_1, g_3 \circ_{\mathscr{D}} g_2 \circ_{\mathscr{D}} g_1), \\ &= (f_3, g_3) \circ_{\mathscr{C}\times\mathscr{D}} (f_2 \circ_{\mathscr{C}} f_1, g_2 \circ_{\mathscr{D}} g_1), \\ &= (f_3, g_3) \circ_{\mathscr{C}\times\mathscr{D}} ((f_2, g_2) \circ_{\mathscr{C}\times\mathscr{D}} (f_1, g_1)). \end{aligned}$$

Hence $\circ_{\mathscr{C}\times\mathscr{D}}$ is associative.

We now show the identities indeed act as identities: Given $(C, D) \in \mathrm{Ob}(\mathscr{C} \times \mathscr{D})$, and $(f_1, g_1) \in \hom_{\mathscr{C}\times\mathscr{D}}((C_1, D_1), (C, D))$, and $(f_2, g_2) \in \hom_{\mathscr{C}\times\mathscr{D}}((C, D), C_2, D_2)$ we have;

$$\begin{aligned} id_{(C,D)} \circ_{\mathscr{C}\times\mathscr{D}} (f_1, g_1) &= (id_C, id_D) \circ_{\mathscr{C}\times\mathscr{D}} (f_1, g_1) \\ &= (id_C \circ_{\mathscr{C}} f_1, id_D \circ_{\mathscr{D}} g_1) \\ &= (f_1, g_1) \end{aligned}$$

and,

$$\begin{aligned} (f_2, g_2) \circ_{\mathscr{C}\times\mathscr{D}} id_{(C,D)} &= (f_2, g_2) \circ_{\mathscr{C}\times\mathscr{D}} (id_C, id_D) \\ &= (f_2 \circ_{\mathscr{C}} id_C, g_2 \circ_{\mathscr{D}} id_D) \\ &= (f_2, g_2). \end{aligned}$$

Therefore $id_{(C,D)}$ acts as an identity with respect to composition. Hence $\mathscr{C} \times \mathscr{D}$ is a category.

The following Definition 2.13 and Lemma 2.14 are adapted from Bartosz Milewsik video lectures [7] and the book by Adámek - Herrlick - Strecker [1] page 22 onward.

Definition 2.13. Given a category $\mathscr{C}$ we define $\mathscr{C}^{op} = (\mathrm{Ob}(\mathscr{C}), \hom_{op}, \circ^{op}, id)$:

1. The objects of $\mathscr{C}^{op}$ are the objects of $\mathscr{C}$;
2. Given $A, B \in \mathrm{Ob}(\mathscr{C})$ for each $f \in \hom(A, B)$ we have a $f^{op} \in \hom_{\mathscr{C}^{op}}(B, A)$;
3. Given $f^{op} \in \hom^{op}(A, B)$ and $g^{op} \in \hom^{op}(B, C)$,

$$g^{op} \circ^{op}_{(A,B,C)} f^{op} = (f \circ_{(C,B,A)} g)^{op}.$$

Lemma 2.14. $\mathscr{C}^{op}$ defined in Definition 2.13 is a category.

Proof. We first prove associativity of the composition for $\mathscr{C}^{op}$; given morphisms $f^{op} \in \hom^{op}(A, B)$, $g^{op} \in \hom^{op}(B, C)$ and $h^{op} \in \hom^{op}(C, D)$ we have,

$$\begin{aligned} h^{op} \circ^{op} (g^{op} \circ^{op} f^{op}) &= h^{op} \circ^{op} (f \circ g)^{op} \\ &= f \circ g \circ h \\ &= f \circ (g \circ h) \\ &= (g \circ h)^{op} \circ^{op} f^{op} \\ &= (h^{op} \circ^{op} g^{op}) \circ^{op} f^{op}. \end{aligned}$$

Hence $\circ^{op}$ is associative.

Now we show the identities hold; given morphisms $f^{op} \in \hom^{op}(A, B)$ and $g^{op} \in \hom^{op}(B, C)$ we have,

$$f^{op} \circ^{op} id_A = (id_A \circ f)^{op} = f^{op}$$

and,

$$id_C \circ^{op} g^{op} = (g \circ id_C)^{op} = g^{op}.$$

Hence each id_C acts as an identity. Therefore, $\mathscr{C}^{op}$ is a category. □

Remark 2.15. Since we can construct a dual category for any category, $\mathscr{C}$ every result we prove about a general category gives us a dual result by looking from the perspective of the dual category.

2.1.2 Example: Monoid as a category

This section follows ideas from the video lectures of Bartosz Milewski [7]. We can define a monoid in terms of the category structure as follows.

Definition 2.16 (Monoid as a category)**.** A *cat monoid* is a category $\mathscr{M}$ with one object, $m \in \text{Ob}(\mathscr{M})$.

We will show that each cat monoid is a monoid and each monoid is a cat monoid and so the definitions are equivalent.

Lemma 2.17. Let $\mathscr{M}$ be a cat monoid with object $m \in \text{Ob}(\mathscr{M})$. Then $(\hom(m,m), \circ, id_m)$ is a monoid.

Proof. $\hom(m,m)$ is a set by Definition 2.1.

For any $g, f \in \hom(m,m)$ we have $g \circ f \in \hom(m,m)$ and so $\circ$ is a closed binary operation.

For each $g \in \hom(m,m)$ we have,

$$id_m \circ g = g$$

and,

$$g \circ id_m = g.$$

$\circ$ is associative since $\mathscr{M}$ is a category.

Hence, $(\hom(m,m), \circ, id_m)$ is a monoid. □

Definition 2.18. Let $M = (M_s, *, e_M)$ be a monoid. We define the category $\mathscr{M}$ as:

1. $\text{Ob}(\mathscr{M})$ has one object m;
2. $M_s = \hom_{\mathscr{M}}(m,m)$;
3. $\circ_{m,m,m} = *$;
4. $id_m = e_M$.

Lemma 2.19. $\mathscr{M}$ as defined above in Definition 2.18 is a cat monoid.

Proof. Firstly, $\circ$ is associative since $*$ is associative as M is a monoid.

We also have for each $f, g \in \hom(m,m)$,

$$f \circ id_m = f * e_M = f$$

and

$$id_m \circ g = e_M * g = g.$$

Therefore $\mathscr{M}$ is a one object category, hence a cat monoid. □

Remark 2.20. Any group can also be seen as a one object category since a group is a special case of a monoid as follows:

Suppose $\mathscr{G}$ is a group as a one object category then we have for each $f \in \hom_{\mathscr{G}}(g,g)$ a morphism $g \in \hom_{\mathscr{G}}(g,g)$ such that,

$$f \circ g = id_g$$

and,

$$g \circ f = id_g.$$

2.1.3 Example: Category of rings

Here we define what a ring is and how it fits into the framework of a category.

Definition 2.21. A *ring with 1* is a triple $R = (R_s, +, \times)$ where R_s is a set called the *underlying set of R*, and $+, \times$ are closed binary operations on R_s called *addition* and *multiplication* respectively.

We also have that the following conditions are satisfied:

1. $(R, +)$ forms an abelian group with identity denoted 0_R;
2. $(R, \times, 1_R)$ forms a monoid with identity $1_R \neq 0_R$;
3. Multiplication is distributive across addition. Given $, y, z \in R_s$,

$$x \times (y + z) = x \times y + x \times z$$

and,

$$(y + z) \times x = y \times x + z \times x.$$

Definition 2.22. An *abelian group* $(G, +)$ is a group with the *commutativity property* that is: Given $x, y \in G$,

$$x + y = y + x.$$

Definition 2.23. A *commutative ring with 1* is a ring with 1 $R = (R_s, +, \times)$ where multiplication is commutative, that is:
Given $x, y \in R_s$,

$$x \times y = y \times x.$$

Definition 2.24. Given two rings with 1, $R = (R_s, +_R, \times_R)$ and $S = (S_S, +_S, \times_S)$ a function $f \colon R_S \to S_S$ is called a *ring homomorphism* if given $x, y \in R_s$ the following axioms hold:

1. $f(x +_R y) = f(x) +_S f(y)$;
2. $f(x \times_R y) = f(x) \times_S f(y)$;
3. $f(1_R) = 1_S$.

Corollary 2.25. Let R and S be a rings with 1 and $f \colon R_S \to S_S$ be a ring homomorphism then:

$$f(0_R) = 0_S.$$

Proof.

$$\begin{aligned} f(1_R) &= f(1_R +_R 0_R) \\ &= f(1_R) +_S f(0_R) \\ &= 1_S +_S f(0_R) \\ &= 1_S, \end{aligned}$$

hence, $f(0_R) = 0_S$. □

Remark 2.26. A commutative ring homomorphism is a ring homomorphism between commutative rings with 1.

Definition 2.27. Define the quadruple $\mathbf{Rng} = (\mathrm{Ob}(\mathbf{Rng}), \hom_{\mathbf{Rng}}, \circ, id)$ where:

1. $\text{Ob}(\mathbf{Rng})$ is the class of all rings with 1;
2. Given $R, S \in \text{Ob}(\mathbf{Rng})$, $\text{hom}_{\mathbf{Rng}}(R, S)$ is the set of all ring homomorphisms between R and S;
3. $\circ$ is regular function composition;
4. Given $R \in \text{Ob}(\mathbf{Rng})$, id_R is the identity with respect to function composition. id_R is a ring homomorphism since for each $x, y \in R$,

$$id_R(x +_R y) = x +_R y = id_R(x) +_R id_R(y)$$

and,

$$id_R(x \times_R y) = x \times_R y = id_R(x) \times_R id_R(y)$$

and,

$$id_R(1_R) = 1_R.$$

Definition 2.28. Define the quadruple $\mathbf{CRng} = (\text{Ob}(\mathbf{Rng}), \text{hom}_{\mathbf{CRng}}, \circ, id)$ where:

1. $\text{Ob}(\mathbf{CRng})$ is the class of all rings with 1;
2. Given $R, S \in \text{Ob}(\mathbf{CRng})$, $\text{hom}_{\mathbf{CRng}}(R, S)$ is the set of all ring homomorphisms between R and S;
3. $\circ$ is regular function composition;
4. Given $R \in \text{Ob}(\mathbf{CRng})$, id_R is the identity with respect to function composition.

   id_R is a commutative ring homomorphism since for each $x, y \in R$,

$$id_R(x +_R y) = x +_R y = id_R(x) +_R id_R(y)$$

and,

$$id_R(x \times_R y) = x \times_R y = id_R(x) \times_R id_R(y)$$

and,

$$id_R(1_R) = 1_R.$$

Lemma 2.29. Both $\mathbf{Rng}$ and $\mathbf{Crng}$ as defined in Definition 2.27 and Definition 2.28 are categories.

Proof. We prove the category $\mathbf{Rng}$, the proof for $\mathbf{Crng}$ follows trivially.

Firstly, $\circ$ is associative since regular function composition is associative.

Given each $R \in \text{Ob}(\mathbf{Rng})$, id_R acts as an identity; Given $f \in \text{hom}_{\mathbf{Rng}}(R, S)$ and $g \in \text{hom}_{\mathbf{CRng}}(S, R)$

$$f \circ id_R = f$$

and,

$$id_S \circ g = g.$$

□

Some other well know categories can be found in Adámek - Herrlick - Strecker [1] and include: $\mathbf{Top}_C$; objects are topological spaces and morphisms are continuous maps or $\mathbf{Top}_H$; objects are topological spaces and morphisms are homeomorphisms of topological spaces and $\mathbf{Met}$; objects are metric spaces with morphisms continuous maps between metric spaces.

We can now translate some familiar concepts from the theories we are used to and translate them to the language of category theory. First we look at an isomorphism.

Definition 2.30. For a category $\mathscr{C}$ a morphism $f \in \hom(A, B)$ is called an isomorphism if there exists a $g \in \hom(B, A)$ such that $f \circ g = id_B$ and $g \circ f = id_A$.

Theorem 2.31. Let $X, Y \in \mathrm{Ob}(\mathbf{Set})$ and $f \in \hom(X, Y)$ then f is an isomorphism if and only if f is a bijection.

Proof. For any two objects $X, Y \in \mathrm{Ob}(\mathbf{Set})$ suppose a map $f \in \hom(X, Y)$ is a bijection therefore f has an inverse $f^{-1} \in \hom(X, Y)$ where, $f \circ f^{-1} = id_X$ and $f^{-1} \circ f = id_Y$. Hence, every bijection is a category isomorphism.

Suppose $f \in \hom(G, H)$ is a isomorphism then there exists a $g \in \hom(H, G)$ such that $f \circ g = id_H$ then f is a bijection with inverse $f^{-1} = g$. Therefore the set of category isomorphisms is exactly the set of group isomorphisms in $\mathbf{Set}$. □

Theorem 2.32. Let $G, H \in \mathrm{Ob}(\mathbf{Grp})$ and $f \in \hom(G, H)$ then f is an isomorphism if and only if f is a group isomorphism.

Proof. For any two objects $G, H \in \mathrm{Ob}(\mathbf{Grp})$ suppose a morphism $f \in \hom(G, H)$ is a group isomorphism then f is a bijection so has an inverse $f^{-1} \in \hom(H, G)$ where, $f \circ f^{-1} = id_H$ and $f^{-1} \circ f = id_G$. Hence, every group isomorphism is a category isomorphism. Suppose $f \in \hom(G, H)$ is a category isomorphism then there exists a $g \in \hom(H, G)$ such that $f \circ g = id_H$ then f is a bijection with inverse $f^{-1} = g$ hence a group isomorphism. Therefore the set of category isomorphisms is exactly the set of group isomorphisms in $\mathbf{Grp}$. □

2.2 Subcategories

The following definitions come from nCatLab [8].

Definition 2.33. Let $\mathscr{C}$ be a category. Then a *subcategory*, $\mathscr{D} = (\mathrm{Ob}(\mathscr{D}), \hom, \circ, id)$ is defined:

1. $\mathrm{Ob}(\mathscr{D})$ is a subclass of $\mathrm{Ob}(\mathscr{C})$;
2. For each $X, Y \in \mathrm{Ob}(C)$, $\hom_{\mathscr{D}}(X, Y)$ is a subset of $\hom_{\mathscr{C}}(X, Y)$;
3. If $f \in \hom_{\mathscr{D}}(X, Y)$ then $X, Y \in \mathscr{D}$;
4. If $f \in \hom_{\mathscr{D}}(X, Y)$ and $g \in \hom_{\mathscr{D}}(Y, Z)$ then $g \circ f \in \hom_{\mathscr{D}}(X, Z)$;
5. For all $X \in \mathrm{Ob}(\mathscr{D})$, $id_X \in \hom_{\mathscr{D}}(X, X)$.

Remark 2.34. Every subcategory D of C is a category since the composition is associative and we have identities.

Definition 2.35. Let $\mathscr{C}$ be a category and $\mathscr{D}$ be a sub category of $\mathscr{C}$.
We say $\mathscr{D}$ is:

1. A *full* subcategory if for all $X, Y \in \mathrm{Ob}(\mathscr{D})$, if $f \in \hom_{\mathscr{C}}(X, Y)$ then $f \in \hom_{\mathscr{D}}(X, Y)$;
2. A *wide* subcategory if for all $X \in \mathrm{Ob}(\mathscr{C})$, $X \in \mathrm{Ob}(\mathscr{D})$.

The following example followed from a discussion with the supervisor.

Example 2.36. Let $\mathbf{Met_C}$ be the category whose objects are metric spaces and morphisms are continuous maps between spaces. Let $\mathbf{Met_I}$ be the category whose objects are metric spaces and morphism are isometries; given two metric spaces $X, Y \in \mathrm{Ob}(\mathbf{Met_I})$, $f \in \hom_{\mathbf{Met_I}}(X, Y)$ if for $x, y \in X$,

$$d_X(x, y) = d_Y(f(x), f(y)).$$

Then $\mathbf{Met_I}$ is a wide, not full, subcategory of $\mathbf{Met_C}$, since every isometry is a continuous map but not every continuous map is an isometry.

Example 2.37. Recall the categories $\mathbf{CRng}$ and $\mathbf{Rng}$ defined in Subsection 2.1.3. $\mathbf{CRng}$ is a full but not wide subcategory of $\mathbf{Rng}$, since every commutative ring is a ring but there exist rings that are not commutative but we have for two commutative rings $R, S \in \mathrm{Ob}(\mathbf{CRng})$ if $f \in \hom_{\mathbf{CRng}}(R, S)$ then $f \in \hom_{\mathbf{Rng}}(R, S)$.

3 Functors

Definitions and examples in this section come from various references including Leinster [5], Adámek - Herrlick - Strecker [1] can define a notion of morphisms between categories as follows.

Definition 3.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *functor* is a pair $F = (F^{ob}, F^{hom})\colon \mathscr{C} \to \mathscr{D}$ where:

1. F^{ob} is a function,
$$F^{ob}\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D});$$
2. For each $A, A' \in \mathrm{Ob}(\mathscr{C})$, F^{hom} is a function,
$$F^{hom}\colon\ \hom_{\mathscr{C}}(A, A') \to \hom_{\mathscr{D}}(F^{ob}(A), F^{ob}(A')),$$
$$f \mapsto F^{hom}(f);$$

The following axioms are satisfied:

1. For all $f \in \hom(A, A')$ and $f' \in \hom(A', A'')$,
$$F^{hom}(f' \circ^{\mathscr{C}} f) = F^{hom}(f') \circ^{\mathscr{D}} F^{hom}(f)$$
where $\circ^{\mathscr{C}}$ and $\circ^{\mathscr{D}}$ are the compositions for $\mathscr{C}, \mathscr{D}$ respectively;
2. For all $A \in \mathrm{Ob}(\mathscr{C})$,
$$F^{hom}(id_A) = id_{F^{ob}(A)}.$$

Remark 3.2. We will just write F for both F^{ob} and F^{hom} and know which one is being used by what object it is acting on.

Definition 3.3. Given a category $\mathscr{C}$ let $F\colon \mathscr{C} \to \mathscr{C}$ be a functor from $\mathscr{C}$ to $\mathscr{C}$, then we call F an *endofunctor*.

3.1 Examples

3.1.1 Forgetful functor for monoids

One of the easiest examples of a functor is the forgetful functor which, informally, takes any category where the objects are sets with added structure and 'forgets' any extra structure. These types of functors play an important role in the theory later in Section 5 and Section 7. We will see the case for monoids.

Definition 3.4 (Forgetful functor for monoids)**.** Let **Mon** be the category of monoids as defined in Lemma 2.11. We define $U\colon \mathbf{Mon} \to \mathbf{Set}$ as:

1. $$U^0\colon \mathrm{Ob}(\mathbf{Mon}) \to \mathrm{Ob}(\mathbf{Set})$$
$$(G, *, e_G) \mapsto G;$$

2. $$U^1\colon\ \hom_{\mathbf{Set}}(G, H) \to \hom_{\mathbf{Mon}}(U(G), U(H))$$
$$f \mapsto f.$$

Lemma 3.5. The $U\colon \mathbf{Mon} \to \mathbf{Set}$ defined above in Definition 3.4 is a functor.

Proof. Let U be the forgetful functor for monoids, then:

1. For all $f \in \hom(G, H)$ and $g \in \hom(H, J)$,
$$U(h \circ^{Mon} g) = U(h) \circ^{Set} U(g)$$
since $U(g) = g$, $U(h) = h$ and $\circ^{Mon}$ is the same as $\circ^{Set}$;
2. For all $G \in \mathrm{Ob}(\mathbf{Mon})$,
$$U(id_G) = id_G = id_{U(G)}.$$

Hence U is a functor. □

3.1.2 Free monoid functor

Informally, free functors take a set and aim to add structure to form a different algebraic object. For example we will give the free functor for monoids, adapted from nCatLab [8]. Free functors seem to be doing the opposite of forgetful functors, we will later make this notion rigorous in Section 5 where these functors are 'adjoint'.

Definition 3.6. Let X be a set. The *free monoid on* X is the triple $(\gamma(X), *, \varnothing)$ where:
$$\gamma(X) = \{(x_1, x_2, \ldots, x_n) | n \in \mathbb{Z}^+, x_1, x_2, \ldots, x_n \in X\} \cup \{\varnothing\}.$$
The elements of $\gamma(X)$ are called *lists* in X.
For any two lists $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_m) \in \gamma(X)$,
$$x * y = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$$
and $*$ is called *concatenation*.
$\varnothing = ()$ is defined as the list with no elements.

Lemma 3.7. For any set X the free monoid on X is a monoid.

Proof. Let X be a set and $(\gamma(X), *, \varnothing)$ the free monoid on X then for any three lists $x, y, z \in \gamma(X)$ we have,
$$\begin{aligned} x * (y * z) &= (x_1, x_2, \ldots, x_n) * ((y_1, y_2, \ldots, y_m) * (z_1, z_2, \ldots, z_k)) \\ &= (x_1, x_2, \ldots, x_n) * (y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_k) \\ &= (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_k) \\ &= (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) * (z_1, z_2, \ldots, z_k) \\ &= ((x_1, x_2, \ldots, x_n) * (y_1, y_2, \ldots, y_m)) * (z_1, z_2, \ldots, z_k) \\ &= (x * y) * z \end{aligned}$$
and,
$$\begin{aligned} x * \varnothing &= (x_1, x_2, \ldots, x_n) * \varnothing \\ &= (x_1, x_2, \ldots, x_n) \\ &= x \\ &= \varnothing * (x_1, x_2, \ldots, x_n) \\ &= \varnothing * x. \end{aligned}$$
Hence concatenation is associative with identity $\varnothing$.
Therefore $(\gamma(X), *, \varnothing)$ is a monoid. □

Lemma 3.8. Given two sets X, Y and a function $f : X \to Y$. There is an induced monoid homomorphism,

$$f_\gamma : (\gamma(X), *, \varnothing) \to (\gamma(Y), *, \varnothing)$$

where for each $(x_1, x_2, \ldots, x_n) \in \gamma(X)$

$$(x_1, x_2, \ldots, x_n) \mapsto (f(x_1), f(x_2), \ldots, f(x_n)),$$

and,

$$\varnothing \mapsto \varnothing.$$

Proof. Let f_γ be defined as above then given any $x, y \in (\gamma(X), *, \varnothing)$,

$$\begin{aligned} f_\gamma(x * y) &= f_\gamma((x_1, x_2, \ldots, x_n) * (y_1, y_2, \ldots, y_m)) \\ &= f_\gamma((x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)) \\ &= (f(x_1), f(x_2), \ldots, f(x_n), f(y_1), f(y_2), \ldots, f(y_m)) \\ &= (f(x_1), f(x_2), \ldots, f(x_n)) * (f(y_1), f(y_2), \ldots, f(y_m)) \\ &= f_\gamma((x_1, x_2, \ldots, x_n)) * f_\gamma((y_1, y_2, \ldots, y_m)) \\ &= f_\gamma(x) * f_\gamma(y) \end{aligned}$$

and,

$$f_\gamma(\varnothing) = \varnothing.$$

Hence f_γ is a monoid homomorphism. □

Definition 3.9. Define the *free monoid functor* $F\colon \mathbf{Set} \to \mathbf{Mon}$ as:

1. For each $X \in \mathrm{Ob}(\mathbf{Set})$,

$$\begin{aligned} F\colon \mathrm{Ob}(\mathbf{Set}) &\to \mathrm{Ob}(\mathbf{Mon}), \\ X &\mapsto (\gamma(X), *, \varnothing); \end{aligned}$$

2. For any $X, Y \in \mathrm{Ob}(\mathbf{Set})$,

$$\begin{aligned} F\colon \hom_{\mathbf{Set}}(X, Y) &\to \hom_{\mathbf{Mon}}(F(X), F(Y)), \\ f &\mapsto f_\gamma. \end{aligned}$$

Lemma 3.10. $F\colon \mathbf{Set} \to \mathbf{Mon}$ defined above in Definition 3.9 is a functor.

Proof. For any $X, Y, Z \in \mathrm{Ob}(\mathbf{Set})$, let $f \in \hom(X, Y)$ and $g \in \hom(Y, Z)$. Then for any list $(x_1, x_2, \ldots, x_n) \in \gamma(X)$ we have,

$$\begin{aligned} F(g \circ f)(x_1, x_2, \ldots, x_n) &= (g \circ f(x_1), g \circ f(x_2), \ldots, g \circ f(x_n)) \\ &= F(g)(f(x_1), f(x_2), \ldots, f(x_n)) \\ &= (F(g) \circ F(f))(x_1, x_2, \ldots, x_n). \end{aligned}$$

For any $X \in \mathrm{Ob}(\mathbf{Set})$, id_X is the identity with respect to function composition.

$$\begin{aligned} F(id_X)(x_1, x_2, \ldots, x_n) &= (id_X(x_1), id_X(x_2), \ldots, id_X(x_n)) \\ &= (x_1, x_2, \ldots, x_n) \\ &= id_{F(X)}(x_1, x_2, \ldots, x_n). \end{aligned}$$

Therefore, F is a functor. □

3.1.3 The Yoneda Embeddings h_A and h^B

An important functor we will use later in Section 5 is the hom functor. This definition is adapted from the Wikipedia article [14]. The following definition is similar to the definition of functors in Definition 3.1 however the morphisms and composition are reversed. See Wiki [13] for more details.

Definition 3.11. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *contravariant functor* is a pair $F = (F^{ob}, F^{hom})\colon \mathscr{C} \to \mathscr{D}$ where:

1. F^{ob} is a function,
$$F^{ob}\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D});$$
2. For each $A, A' \in \mathrm{Ob}(\mathscr{C})$, F^{hom} is a function,
$$\begin{aligned} F^{hom}\colon\ \hom_{\mathscr{C}}(A, A') &\to \hom_{\mathscr{D}}(F^{ob}(A'), F^{ob}(A)), \\ f &\mapsto F^{hom}(f); \end{aligned}$$

The following axioms are satisfied:

1. For all $f \in \hom(A, A')$ and $f' \in \hom(A', A'')$,
$$F^{hom}(f' \circ^{\mathscr{C}} f) = F^{hom}(f') \circ^{\mathscr{D}} F^{hom}(f)$$
where $\circ^{\mathscr{C}}$ and $\circ^{\mathscr{D}}$ are the compositions for $\mathscr{C}, \mathscr{D}$ respectively;
2. For all $A \in \mathrm{Ob}(\mathscr{C})$,
$$F^{hom}(id_A) = id_{F^{ob}(A)}.$$

Lemma 3.12. Let $\mathscr{C}$ be a category and **Set** be the category of sets as in Example 2.4.

1. For all $A \in \mathrm{Ob}(\mathscr{C})$ we define the functor
$$\mathrm{h_A}\colon \mathscr{C} \to \mathbf{Set}$$
where for each $C \in \mathrm{Ob}(\mathscr{C})$,
$$C \mapsto \hom_{\mathscr{C}}(A, C)$$
and for each $f \in \hom_{\mathscr{C}}(X, Y)$,
$$f \mapsto \hom(A, f)$$
where,
$$\begin{aligned} \hom_{\mathscr{C}}(A, f)\colon\ \hom_{\mathscr{C}}(A, X) &\to \hom_{\mathscr{C}}(A, Y), \\ g &\mapsto f \circ g; \end{aligned}$$
2. For all $B \in \mathrm{Ob}(\mathscr{C})$ we define the *contravariant* functor
$$\mathrm{h^B}\colon \mathscr{C}^{op} \to \mathbf{Set}$$
where for each $C \in \mathrm{Ob}(\mathscr{C})$,
$$C \mapsto \hom_{\mathscr{C}}(C, B)$$
and for each $h \in \hom_{\mathscr{C}^{op}}(X, Y)$,
$$h \mapsto \hom(h, B),$$
where,
$$\begin{aligned} \hom(h, B)\colon\ \hom_{\mathscr{C}}(Y, B) &\to \hom_{\mathscr{C}}(X, B), \\ g &\mapsto g \circ h. \end{aligned}$$

Proof. First we prove h_A is a functor.

Let $C \in \mathrm{Ob}(\mathscr{C})$ then given $f \in \hom_{\mathscr{C}}(A, C)$,

$$\begin{aligned}\mathrm{h}_A(id_C)(f) &= \hom_{\mathscr{C}}(A, id_C)(f) \\ &= f \\ &= id_{\hom_{\mathscr{C}}(A,C)}(f).\end{aligned}$$

Hence identities are preserved. We also have for any $f \in \hom_{\mathscr{C}}(X, Y)$ and $g \in \hom_{\mathscr{C}}(Y, Z)$ then given $h \in \hom_{\mathscr{C}}(A, X)$,

$$\begin{aligned}\mathrm{h}_A(g \circ f)(h) &= \hom_{\mathscr{C}}(A, g \circ f)(h) \\ &= g \circ f \circ h \\ &= \hom_{\mathscr{C}}(A, g) \circ \hom_{\mathscr{C}}(A, f)(h) \\ &= (\mathrm{h}_A(g) \circ \mathrm{h}_A(f))(h).\end{aligned}$$

Hence composition is preserved. Therefore h_A is a functor.

Now we show h^B is a contravariant functor. Let $C \in \mathrm{Ob}(\mathscr{C})$ then given $f \in \hom_{\mathscr{C}}(C, B)$:

$$\begin{aligned}\mathrm{h}^B(id_C)(f) &= \hom_{\mathscr{C}}(id_C, B)(f) \\ &= f \\ &= id_{\hom_{\mathscr{C}}(C,B)}(f).\end{aligned}$$

Hence identities are preserved. We also have for any $f \in \hom_{\mathscr{C}}(X, Y)$ and $g \in \hom_{\mathscr{C}}(Y, Z)$ then given $h \in \hom_{\mathscr{C}^{op}}(Z, B)$,

$$\begin{aligned}\mathrm{h}^B(g \circ f)(h) &= \hom_{\mathscr{C}}(g \circ f, B)(h) \\ &= h \circ g \circ f \\ &= \hom_{\mathscr{C}}(f, B) \circ \hom_{\mathscr{C}}(g, B)(h) \\ &= (\mathrm{h}^B(f) \circ \mathrm{h}^B(g))(h).\end{aligned}$$

Hence composition is preserved. Therefore h^B is a contravariant functor. □

Definition 3.13. Let $\mathscr{C}$ be a category then $\mathscr{C}^{op} \times \mathscr{C}$ is a category by Definition 2.13 and Example 2.12. We define the *hom functor*, $\mathrm{h}\colon \mathscr{C}^{op} \times \mathscr{C} \to \mathbf{Set}$, as follows: Given $(C_1, C_2) \in \mathrm{Ob}(\mathscr{C}^{op} \times \mathscr{C})$,

$$(C_1, C_2) \mapsto \hom_{\mathscr{C}}(C_1, C_2).$$

For each $f^{op} \in \hom_{\mathscr{C}}(C_1, C_2)$ and $g \in \hom_{\mathscr{C}}(C_1', C_2')$,

$$(f^{op}, g) \mapsto \hom_{\mathscr{C}}(f, g),$$

where,

$$\begin{aligned}\hom_{\mathscr{C}}(f, g)\colon \hom_{\mathscr{C}}(C_1, C_1') &\to \hom_{\mathscr{C}}(C_2, C_2'), \\ h &\mapsto g \circ h \circ f.\end{aligned}$$

Lemma 3.14. h as defined in Definition 3.13 is a functor.

Proof. Let $(f_1^{op}, g_1) \in \hom_{\mathscr{C}^{op}\times\mathscr{C}}((C_1, C_2), (C_1', C_2'))$, and $(f_2^{op}, g_2) \in \hom_{\mathscr{C}^{op}\times\mathscr{C}}((C_1', C_2'), (C_1'', C_2''))$, we then have,

$$\begin{aligned}\mathrm{h}((f_1^{op}, g_1) \circ (f_2^{op}, g_2)) &= \mathrm{h}((f_1^{op} \circ f_2^{op}, g_1 \circ g_2)), \\ &= \hom_{\mathscr{C}}(f_1^{op} \circ^{op} f_2^{op}, g_1 \circ g_2),\end{aligned}$$

where $\hom_{\mathscr{C}}(f_2 \circ f_1, g_1 \circ g_2)$ is the function which takes $h \in \hom_{\mathscr{C}}(C_1, C_1')$,

$$h \mapsto g_1 \circ g_2 \circ f_2 \circ f_1.$$

Hence is the same as the function,

$$\hom_{\mathscr{C}}(f_1, g_1) \circ \hom_{\mathscr{C}}(f_2, g_2) = \mathrm{h}((f_1^{op}, g_1)) \circ \mathrm{h}((f_2^{op}, g_2)).$$

Given $(id_{C_1}, id_{C_2}) \in \hom_{\mathscr{C}^{op} \times \mathscr{C}}((C_1, C_2), (C_1, C_2))$ we have,

$$\mathrm{h}((id_{C_1}, id_{C_2})) = \hom_{\mathscr{C}}(id_{C_1}, id_{C_2}),$$

which is the function which sends $h \in \hom_{\mathscr{C}}(C_1, C_1')$

$$h \mapsto id_{C_1} \circ h \circ id_{C_2} = h.$$

Therefore, h is a functor. □

3.1.4 The Identity functor

For every category there exists a functor from that category to itself called the identity functor we will now define.

Definition 3.15. For any category $\mathscr{C}$ let $id_{\mathscr{C}} \colon \mathscr{C} \to \mathscr{C}$ be the identity functor on $\mathscr{C}$ defined:

1. For each $X \in \mathrm{Ob}(\mathscr{C})$,
$$id_{\mathscr{C}}(X) = X.$$

2. Given two $\mathscr{C}$-Objects, $X, Y \in \mathrm{Ob}(\mathscr{C})$, for each $f \in \hom(X, Y)$,
$$id_{\mathscr{C}}(f) = f.$$

Lemma 3.16. $id_{\mathscr{C}}$ as defined above in Definition 3.15 is a functor.

Proof. Given any $X, Y, Z \in \mathscr{C}$, let $f \in \hom(X, Y)$ and $g \in \hom(Y, Z)$ be maps. Then,

$$\begin{aligned} id_{\mathscr{C}}(f \circ g) &= f \circ g \\ &= id_{\mathscr{C}}(f) \circ id_{\mathscr{C}}(g). \end{aligned}$$

Hence composition is preserved.

For all $X \in \mathrm{Ob}(\mathscr{C})$

$$id_{\mathscr{C}}(id_X) = id_X.$$

Hence identities are preserved. Therefore $id_{\mathscr{C}}$ is a functor. □

3.1.5 More examples of functors

Here are some more examples of functors.

The following example followed from a meeting with the supervisor.

Example 3.17. Let the power set functor P : **Set** $\to$ **Set** be defined:

1. For each $X \in \mathrm{Ob}(\mathbf{Set})$,
$$X \mapsto \mathcal{P}(X)$$
where $\mathcal{P}(X)$ is the power set of X,
$$\mathcal{P}(X) = \{A \mid A \subseteq X\};$$

2. Given two **Set**-Objects $X, Y \in \mathrm{Ob}(\mathbf{Set})$ then for each $f \in \hom(X, Y)$,
$$\begin{aligned} P(f) : P(X) &\to P(Y), \\ A &\mapsto f[A] \end{aligned}$$
where $f[A]$ is the image of A under f,
$$f[A] = \{f(a) \mid a \in A\}.$$
Clearly, $P(f)$ is a function $P(X) \to P(Y)$, so it is in $\hom(P(X), P(Y))$.

Given any $X, Y, Z \in \mathbf{Set}$, let $f \in \hom(X, Y)$ and $g \in \hom(Y, Z)$ be functions. Then for all $A \in P(X)$,
$$\begin{aligned} P(g \circ f)(A) &= (g \circ f)[A]) \\ &= g[f[A]] \\ &= P(g) \circ P(f)(A). \end{aligned}$$

For each $X \in \mathbf{X}$, id_X is the identity function with respect to function composition. For all $A \in P(X)$,
$$\begin{aligned} P(id_X)(A) &= id_X[A] \\ &= A \\ &= id_{P(X)}(A) \end{aligned}$$

Therefore P is a functor.

The following Example 3.18 uses ideas from the Wikipedia article [15].

Example 3.18. Let $\mathrm{GL}_2 \colon \mathbf{CRng} \to \mathbf{Grp}$ be defined:

1. For each **CRng**-Object $R \in \mathrm{Ob}(\mathbf{CRng})$,
$$\mathrm{GL}_2(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in R, ab - cd \text{ is invertable in } R \right\}$$
$\mathrm{GL}_2(R) \in \mathbf{Grp}$ since each matrix is invertable, the matrix:
$$\begin{bmatrix} 1_R & 0_R \\ 0_R & 1_R \end{bmatrix}$$
acts as an identity under matrix multiplication and matrix multiplication is associative.

2. Given $R, S \in \mathrm{Ob}(\mathbf{CRng})$ then for each $f \in \hom(R, S)$,
$$\begin{aligned} \mathrm{GL}_2(f) \colon \mathrm{GL}_2(R) &\to \mathrm{GL}_2(S), \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &\mapsto \begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix}. \end{aligned}$$

Then for $M, N \in \mathrm{GL}_2(R)$ we have,

$$\begin{aligned}
\mathrm{GL}_2(f)(M \times_{\mathrm{GL}_2(R)} N) &= \mathrm{GL}_2(f)\left(\begin{bmatrix} a_M \times_R a_N +_R b_M \times_R c_N & a_N \times_R b_M +_R b_N \times_R d_N \\ c_M \times_R a_N +_R d_M \times_R c_N & c_M \times_R b_N +_R d_M \times d_N \end{bmatrix}\right) \\
&= \begin{bmatrix} f(a_M \times_R a_N +_R b_M \times_R c_N) & f(a_N \times_R b_M +_R b_N \times_R d_N) \\ f(c_M \times_R a_N +_R d_M \times_R c_N) & f(c_M \times_R b_N +_R d_M \times_R d_N) \end{bmatrix} \\
&= \begin{bmatrix} f(a_M) \times_S f(a_N) +_S f(b_M) \times_S f(c_N) & f(a_N) \times_S f(b_M) +_S f(b_N) \times_S f(d_N) \\ f(c_M) \times_S f(a_N) +_S f(d_M) \times_S f(c_N) & f(c_M) \times_S f(b_N) +_S f(d_M) \times_S f(d_N) \end{bmatrix} \\
&= \mathrm{GL}_2(f)(M) \times_{\mathrm{GL}_2(S)} \mathrm{GL}_2(f)(N)
\end{aligned}$$

and,

$$\begin{aligned}
\mathrm{GL}_2(f)(1_{\mathrm{GL}_2(R)}) &= \mathrm{GL}_2(f)\left(\begin{bmatrix} 1_R & 0_R \\ 0_R & 1_R \end{bmatrix}\right) \\
&= \begin{bmatrix} f(1_R) & f(0_R) \\ f(0_R) & f(1_R) \end{bmatrix} \\
&= \begin{bmatrix} 1_S & 0_S \\ 0_S & 1_S \end{bmatrix}, & \text{since f is a ring homomorphism} \\
&= 1_{\mathrm{GL}_2(S)}.
\end{aligned}$$

Therefore $\mathrm{GL}_2(f)$ is defined as a group homomorphism.
We now show GL_2 is a functor by proving the axioms hold. Given any $R, S, T \in \mathrm{Ob}(\mathbf{CRng})$, let $f \in \hom(R, S)$ and $g \in \hom(S, T)$ be ring homomorphisms. Then for all

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(R)$$

we have,

$$\begin{aligned}
\mathrm{GL}_2(g \circ f)(M) &= \begin{bmatrix} (g \circ f)(a) & (g \circ f)(b) \\ (g \circ f)(c) & (g \circ f)(d) \end{bmatrix} \\
&= \begin{bmatrix} g(f(a)) & g(f(b)) \\ g(f(c)) & g(f(d)) \end{bmatrix} \\
&= \mathrm{GL}_2(g)\left(\begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix}\right) \\
&= (\mathrm{GL}_2(g) \circ \mathrm{GL}_2(f))(M)
\end{aligned}$$

and,

$$\begin{aligned}
\mathrm{GL}_2(id_R)(M) &= \begin{bmatrix} id_R(a) & id_R(b) \\ id_R(c) & id_R(d) \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= M \\
&= id_{\mathrm{GL}_2(R)}(M).
\end{aligned}$$

Therefore GL_2 is a functor.

Remark 3.19. Example 3.18 can be extended to GL_n with $n \times n$ matrices.

Lemma 3.20. Let R be a ring with 1. Let R^* be the set of all $r \in R$ such that,

$$rr' = r'r = 1_R$$

for some $r' \in R$. Then $(R^*, \times_R)$ forms a group called the *group of units of R*.

Proof. We have an identity 1_R since for each $r \in R^*$,

$$r1_R = r = 1_R r$$

since R is a ring.

Given $r \in R^*$ there exists $r' \in R^*$ such that,

$$rr' = r'r = 1_R.$$

Hence each r has inverse r'.

We know $\times_S$ is associative since R is a ring.

Finally given $r, s \in R^*$,

$$\begin{aligned} rss'r' &= r1_R r' \\ &= rr' \\ &= 1_R \\ &= s's \\ &= s'1_R s \\ &= s'r'rs. \end{aligned}$$

Hence $rs \in R^*$ and therefore R^* is a group. □

Example 3.21. Let $\mathrm{Units}\colon \mathbf{Crng} \to \mathbf{Grp}$ be defined:

1. For each $R \in \mathbf{Crng}$,
$$\mathrm{Units}(R) = R^*$$
as in Lemma 3.20;
2. Given $R, S \in \mathrm{Ob}(\mathbf{Crng})$, for each $f \in \hom(R, S)$,
$$\begin{aligned} \mathrm{Units}(f)\colon R^* &\to S^*, \\ r &\mapsto f(r) \end{aligned}$$
and $f(r) \in S^*$ since,
$$\begin{aligned} 1_S &= f(1_R) \\ &= f(rr') \\ &= f(r)f(r'), && \text{since } f \text{ is a ring homomorphism.} \end{aligned}$$
We also have $\mathrm{Units}(f)$ is a group homomorphism since f is a ring homomorphism.

We will now show that Units is indeed a functor.

Given any $R, S, T \in \mathrm{Ob}(\mathbf{Crng})$, let $f \in \hom(R, S)$ and $g \in \hom(S, T)$. Then for all $r \in R$,

$$\begin{aligned} \mathrm{Units}(g \circ f)(r) &= g \circ f(r) \\ &= \mathrm{Units}(g) \circ \mathrm{Units}(f)(r). \end{aligned}$$

Hence composition is preserved. We also have,

$$\begin{aligned}\text{Units}(id_R)(r) &= id_R(r)\\ &= r\\ &= id_{R^*}(r).\end{aligned}$$

Hence identities are preserved. Therefore Units is a functor.

The following example followed from a lecture on Topology given by Dr Daniel Graves at the University of Leeds.

Example 3.22. Let $\mathbf{Top}^*$ be the category of based topological spaces as defined in Example 2.7 and $\mathbf{Grp}$ be the category of groups as defined in Example 2.5.

The fundamental group functor $\pi_1\colon \mathbf{Top}^* \to \mathbf{Grp}$ be defined:

1. For each $((X, \tau_X), x_0) \in \text{Ob}(\mathbf{Top}^*)$,

$$((X, \tau_X), x_0) \mapsto \pi_1(((X, \tau_X), x_0))$$

   where $\pi_1(((X, \tau_X), x_0))$ is the fundamental group of X based at x_0. The group operation $*$ is defined by join of paths;

2. For a given continuous map $f \in \hom_{\mathbf{Top}^*}(((X, \tau_X), x_0), ((Y, \tau_Y), y_0))$ we define the induced group homomorphism:

$$f \mapsto f^*\colon \pi_1(((X, \tau_X), x_0)) \to \pi_1(((Y, \tau_Y), y_0))$$

   where for each $[\gamma] \in \pi_1(((X, \tau_X), x_0))$,

$$[\gamma] \mapsto [f \circ \gamma].$$

   Here $[\gamma]$ is the equivalence class of a loop γ based at x_0.

   f^* is well defined since for any two loops γ_1 and γ_2 based at x_0 where $[\gamma_1] = [\gamma_2]$ we have a path homotopy,

$$H\colon [0, 1] \times [0, 1] \to X$$

   and therefore,

$$f \circ H\colon [0, 1] \times [0, 1] \to Y,$$

   is a path homotopy between $f \circ \gamma_1$ and $f \circ \gamma_2$ hence f^* is well defined.

   The group identity for $\pi_1(((X, \tau_X), x_0))$ is the equivalence class $[\gamma_{x_0}]$ where γ_{x_0} is the constant path at x_0 so,

$$[f \circ \gamma_{x_0}] = [\gamma_{f(x)}] = [\gamma_{y_0}]$$

   where γ_{y_0} is the constant path at $f(x_0)$. Therefore f^* preserves group identities.

   We also have for any two $[\gamma_1], [\gamma_2] \in \pi_1(((X, \tau_X), x_0))$

$$\begin{aligned}f^*([\gamma_1] * [\gamma_2]) &= f^*([\gamma_1 * \gamma_2])\\ &= [f \circ \gamma_1 * \gamma_2]\\ &= [f \circ \gamma_1 * f \circ \gamma_2]\\ &= [f \circ \gamma_1] * [f \circ \gamma_2]\\ &= f^*([\gamma_1]) * f^*([\gamma_2]).\end{aligned}$$

   Therefore f^* is a group homomorphism.

To check π_1 is a functor we check it preserves identities $id_X \in \hom_{\mathbf{Top}^*}(X, X)$, given a path equivalence class $[\gamma] \in \pi_1(((X, \tau_X), x_0))$.

$$\begin{aligned} \pi_1(id_X)([\gamma]) &= [id_X \circ \gamma] \\ &= [\gamma] \\ &= id_{\pi_1(((X,\tau_X),x_0))}([\gamma]). \end{aligned}$$

Also given $f \in \hom_{\mathbf{Top}^*}(((X, \tau_X), x_0), ((Y, \tau_Y), y_0))$ and $g \in \hom_{\mathbf{Top}^*}(((Y, \tau_Y), y_0), ((Z, \tau_Z), z_0))$ we have,

$$\begin{aligned} (g \circ f)^*([\gamma]) &= [g \circ f \circ \gamma] \\ &= g^*([f \circ \gamma]) \\ &= g^*(f^*([\gamma])) \\ &= g^* \circ f^*([\gamma]). \end{aligned}$$

Therefore π_1 is a functor.

The following example can be found in Leinster [5].

Example 3.23. Recall for each monoid $\mathscr{M}$ we have a one object category defined in subsection 2.1.2, also recall the category **Set** in Example 2.4.

Given a set X, we define a functor

$$\mathrm{L_X}\colon \mathscr{M} \to \mathbf{Set}$$

where for the unique object $M \in \mathscr{M}$,

$$M \mapsto X,$$

and for each morphism $g \in \hom_{\mathscr{M}}(M, M)$,

$$g \mapsto L_X(g)$$

where,

$$\begin{aligned} L_X(g)\colon X &\to X, \\ x &\mapsto gx. \end{aligned}$$

The function $L_X(g)$ is the left action of g on X.

We see for each $x \in X$,

$$\begin{aligned} L_X(id_M)(x) &= id_M x \\ &= x \\ &= id_{L_X(M)} \end{aligned}$$

and for $g, g' \in \hom_{\mathscr{M}}(M, M)$ and each $x \in X$,

$$\begin{aligned} L_X(g' \circ g)(x) &= g' \circ gx \\ &= g'gx \\ &= L_X(g') \circ L_X(g)(x). \end{aligned}$$

Therefore L_X is a functor. For each set $X \in \mathrm{Ob}(\mathbf{Set})$ the functor L_X represents the left action of $\mathscr{M}$ on that set.

3.2 Functor composition

Ideas from this section are adapted from Leinster [5] and Adámek - Herrlick - Strecker [1].

Definition 3.24. Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be categories and $F_1 \colon \mathscr{C} \to \mathscr{D}$, $F_2 \colon \mathscr{D} \to \mathscr{E}$ functors. Then we define $F_1 \circ F_2 \colon \mathscr{C} \to \mathscr{E}$:

1.
$$\begin{aligned}(F_1 \circ F_2)^{ob} \colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{E}),\\ A &\mapsto F_2(F_1(A));\end{aligned}$$

2.
$$\begin{aligned}(F_1 \circ F_2)^{hom} \colon \ \hom_{\mathscr{C}}(A, A') &\to \hom_{\mathscr{E}}((F_1 \circ F_2)^{ob}(A), (F_1 \circ F_2)^{ob}(A'),\\ f &\mapsto F_2(F_1(f)).\end{aligned}$$

Remark 3.25. Again we will drop notation as in Remark 3.2 for simplicity.

Lemma 3.26. $F_1 \circ F_2 \colon \mathscr{C} \to \mathscr{E}$ as defined above is a functor.

Proof. Let $f \in \hom_{\mathscr{C}}(A, A')$ and $g \in \hom_{\mathscr{C}}(A', A'')$ then,

$$\begin{aligned}F_1 \circ F_2(f \circ^{\mathscr{C}} g) &= F_2(F_1(f \circ^{\mathscr{C}} g))\\ &= F_2(F_1(f) \circ^{\mathscr{D}} F_1(g)), && \text{since } F_1 \text{ is a functor}\\ &= F_2(F_1(f)) \circ^{\mathscr{E}} F_2(F_1(g)), && \text{since } F_2 \text{ is a functor}\\ &= F_1 \circ F_2(f) \circ^{\mathscr{E}} F_1 \circ F_2(g).\end{aligned}$$

Given $A \in \mathrm{Ob}(\mathscr{C})$

$$\begin{aligned}F_1 \circ F_2(id_A) &= F_2(F_1(id_A))\\ &= F_2(id_{F_1(A)})\\ &= id_{F_2(F_1(A))}\\ &= id_{F_1 \circ F_2(id_A)}.\end{aligned}$$

Therefore $F_1 \circ F_2$ is a functor. □

Remark 3.27. For functors F, G we will write $FG = F \circ G$ and If F is an endofunctor, then we can write $F^2 = F \circ F$.

4 Natural transformations

Definition 4.1. Let $\mathscr{C}, \mathscr{D}$ be categories. Let $F \colon \mathscr{C} \to \mathscr{D}$, and $G \colon \mathscr{C} \to \mathscr{D}$ be functors. A *natural transformation*, $\eta \colon F \Rightarrow G$ between F and G is an assignment which sends each $\mathscr{C} - Object$, A, to a map $\eta_A \in \hom(F(A), G(A))$, where the *naturality condition* holds:

Given $A, A' \in \mathrm{Ob}(\mathscr{C})$, then for each map $f \in \hom(A, A')$,

$$\eta_{A'} \circ F(f) = G(f) \circ \eta_A. \tag{4.1}$$

Equation 4.1 is equivalent to saying the diagram,

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_{A'}} \\ G(A) & \xrightarrow[G(f)]{} & G(A') \end{array}$$

commutes.

We call η_A *the component of* η *at* A.

We see some examples below.

4.1 Examples

Example 4.2. Let P be the power set functor as in 3.17 and $id_{\mathbf{Set}}$ be the identity functor on **Set** as in 3.15. We have natural transformation, $\eta \colon id_{\mathbf{Set}} \Rightarrow P$. such that, for each $X \in \mathrm{Ob}(\mathbf{Set})$,

$$\begin{aligned} \eta_X \colon id_{\mathbf{Set}}(X) &\to P(X), \\ x &\mapsto \{x\}. \end{aligned}$$

We need to show the naturality condition holds. Let $X, Y \in \mathrm{Ob}(\mathbf{Set})$ and $f \in \hom(X, Y)$. For each $x \in id_{\mathbf{Set}}(X)$,

$$\begin{aligned} (\eta_Y \circ id_{\mathbf{Set}}(f))(x) &= \eta_Y(f(x)) \\ &= \{f(x)\} \\ &= P(f)(\{x\}) \\ &= (P(f) \circ \eta_X)(x). \end{aligned}$$

Therefore the naturality condition holds and η is a natural transformation.

Another example of a natural transformation is given below.

Example 4.3. Let $P \colon \mathbf{Set} \to \mathbf{Set}$ be the power set functor as defined in 3.17 and $P^2 : \mathbf{Set} \to \mathbf{Set}$ be the composition of P with its self as defined in 3.24. Then we can define $\mu \colon P^2 \Rightarrow P$ where, for each $X \in \mathrm{Ob}(\mathbf{Set})$, we have,

$$\begin{aligned} \mu_X \colon P^2(X) &\to P(X), \\ A &\mapsto \bigcup_{I \in A} I. \end{aligned}$$

We need to show the naturality condition holds. Since A is a set of subsets of X then $\bigcup_{I\in A} I$ will be a subset of X. Furthermore for a set A, $\bigcup_{I\in\mathcal{P}(A)} I = A$. Then given $X, Y \in \mathrm{Ob}(\mathbf{Set})$ and $f \in \hom(X, Y)$. For each $A \in P^2(X)$,

$$\begin{aligned}(\mu_Y \circ P^2(f))(A) &= \mu_Y(f[A]) \\ &= \bigcup_{I\in f[A]} I \\ &= P^2(f)(\bigcup_{I\in A} I) \\ &= P^2(f) \circ \mu_X(A).\end{aligned}$$

Where the third equality is true since the union of an image of a set is equal to the image of the union. Therefore we have defined a natural transformation.

These two natural transformations have some importance we discuss later. Here is another example of a natural transformation.

Example 4.4. Recall the functors $\mathrm{GL}_2\colon \mathbf{Crng} \to \mathbf{Grp}$ and $\mathrm{Units}\colon \mathbf{Crng} \to \mathbf{Grp}$ defined in Example 3.18 and Example 3.21 respectively,

Let $\det\colon \mathrm{GL}_2 \Rightarrow \mathrm{Units}$ be defined for each $R \in \mathbf{Crng}$ as,

$$\begin{aligned}\det_R\colon \mathrm{GL}_2(R) &\to R^* \\ \begin{bmatrix} a & b \\ c & d\end{bmatrix} &\mapsto a \times_R d +_R -b \times_R c.\end{aligned}$$

Here $-b$ is the additive inverse of b.

$a \times_R d +_R -b \times_R c \in R^*$ since it is invertable in R.

We first show for each $R \in \mathbf{Crng}$ that $\det_R$ is a group homomorphism. Given $M, N \in \mathrm{GL}_2(R)$, we know $\det(MN) = \det(M)\det(N)$.

We need to show the naturality condition holds. Given $R, S \in \mathbf{CRng}$ and any $f \in \hom(R, S)$ then for each $\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \mathrm{GL}_2(R)$ we have,

$$\begin{aligned}(\mathrm{Units}(f) \circ \det_R)\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right) &= \mathrm{Units}(f)(a \times_R d +_R -b \times_R c) \\ &= f(a \times_R d +_R -b \times_R c) \\ &= f(a) \times_S f(d) +_S -f(b) \times_S f(c) \\ &= \det_S\left(\begin{bmatrix} f(a) & f(b) \\ f(c) & f(d)\end{bmatrix}\right) \\ &= (\det_S \circ \mathrm{GL}_2(f))(\begin{bmatrix} a & b \\ c & d\end{bmatrix}).\end{aligned}$$

The third equality is true since additive inverses are preserved under ring homomorphisms. Hence the naturality condition holds, therefore $\det$ is a natural transformation.

The following example comes from Leinster [5] (Page 29 Example 1.3.4).

Example 4.5. Let $X, Y \in \mathrm{Ob}(\mathbf{Set})$ be sets then recall from Example 3.23 we have functors L_X and L_Y representing left actions of the monoid $\mathscr{M}$ on the sets X and Y respectively. There is a natural transformation

$$\alpha\colon L_X \Rightarrow L_Y$$

where for the single object $m \in \mathrm{Ob}(\mathscr{M})$ we have a map $\alpha_m \in \hom_{\mathbf{Set}}(X, Y)$ where the naturality condition implies for each $g \in \hom_{\mathscr{M}}(M, M)$,

$$\alpha_M \circ L_X(g) = L_Y(g) \circ \alpha_M$$

so for an element $x \in X$,

$$\alpha_M(L_g(x)) = L_Y(g)(\alpha_M(x)),$$

that is,

$$\alpha_M(gx) = g\alpha_M(x),$$

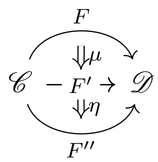
So the natural transformation α represents a map of the sets X, Y preserving left action from $\mathscr{M}$.

4.2 Composition of natural transformations

There are different ways to compose natural transformations; we can take natural transformations with the same domain and co-domain (vertical composition) or we can compose a natural transformation with a functor which leads to an alternative definition of composition (Horizontal composition).

4.2.1 Vertical composition of natural transformations

First we compose two natural transformations whose functors they act on have the same domain and co-domain as demonstrated in the pasting diagram below. Ideas from this section are thanks to Riehl [9] and Leinster [5].



Definition 4.6 (Vertical composition)**.** Let $\mathscr{C}, \mathscr{D}$ be categories, $\mathrm{F}, \mathrm{F}', \mathrm{F}'' \colon C \to D$ be functors and $\eta \colon F \Rightarrow F'$ and $\mu \colon F' \Rightarrow F''$ natural transformations. We then define

$$\mu \cdot \eta \colon F \Rightarrow F''$$

where for each $X \in \mathscr{C}$,

$$(\mu \cdot \eta)_X = \mu_X \circ \eta_X.$$

Remark 4.7. We may also use the notation $(\mu \circ \eta)_X$ to mean $(\mu \cdot \eta)_X$

Lemma 4.8. The vertical composition of two natural transformations as defined above in Definition 4.6 is a natural transformation.

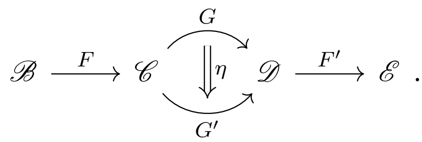
Proof. We need to prove that for $\mu \cdot \eta$ the naturality condition holds. Given $X, Y \in \mathrm{Ob}(\mathscr{C})$ and a map $f \in \hom(X, Y)$ we have,

$$\begin{aligned}
(\mu \cdot \eta)_Y \circ F(f) &= \mu_Y \circ (\eta_Y \circ F(f)) \\
&= \mu_Y \circ (F'(f) \circ \eta_X) && \text{since } \eta \text{ is a natural transformation,} \\
&= (F''(f) \circ \mu_X) \circ \eta_X && \text{since } \mu \text{ is a natural transformation,} \\
&= F''(f) \circ (\mu \cdot \eta)_X.
\end{aligned}$$

There for the naturality condition holds and therefore the composition of two natural transformations is a natural transformation. □

4.2.2 Horizontal composition of natural transformations

First we see how natural transformations can compose with functors. The diagram below illustrates how this should work.



Definition 4.9. Let $\mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories, $\mathrm{F}\colon \mathscr{B} \to \mathscr{C}$, $\mathrm{G}, \mathrm{G}'\colon \mathscr{C} \to \mathscr{D}$ and $\mathrm{F}'\colon \mathscr{D} \to \mathscr{E}$ be functors and $\eta\colon G \Rightarrow G'$ a natural transformation. Given $B \in \mathscr{B}$, we define the natural transformation $\eta F\colon GF \Rightarrow G'F$ where for each $B \in \mathrm{Ob}(\mathscr{B})$,

$$(\eta F)_B = \eta_{F(B)}.$$

Given $C \in \mathscr{C}$ we also define the natural transformation $F'\eta\colon F'G \Rightarrow F'G'$ where for each $C \in \mathrm{Ob}(\mathscr{C})$,

$$(F'\eta)_C = F'(\eta_C).$$

Lemma 4.10. ηF and $F'\eta$ as defined above in Definition 4.9 are natural transformations.

Proof. We need to show that the naturality condition holds. First for ηF. Given $B, B' \in \mathscr{B}$ and $f \in \hom(B, B')$ we have,

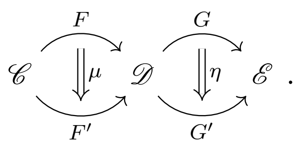
$$\begin{aligned}
(\eta F)_{B'} \circ GF(f) &= \eta_{F(B')} \circ GF(f) \\
&= \eta_{F(B')} \circ G(F(f)) \\
&= G'(F(f)) \circ \eta_{F(B)}, && \text{by the naturality of } \eta \\
&= G'F(f) \circ \eta_{F(B)}.
\end{aligned}$$

Now we prove $F'\eta$ is natural. Given $C, C' \in \mathscr{C}$ and $g \in \hom(C, C')$ we have,

$$\begin{aligned}
(F'\eta)_{C'} \circ F'G(g) &= F'(\eta_{C'}) \circ F'G(g) \\
&= F'(\eta_{C'} \circ G(g)) \\
&= F'(G'(g) \circ \eta_C), && \text{by the naturality of } \eta \\
&= F'G'(g) \circ F'(\eta_C) \\
&= F'G'(g) \circ (F'\eta)_C.
\end{aligned}$$

□

Now we can define horizontal composition of natural transformations, the diagram below illustrates this,



Definition 4.11. Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories, $\mathrm{F}, \mathrm{F}'\colon \mathscr{C} \to \mathscr{D}, \mathrm{G}, \mathrm{G}'\colon \mathscr{D} \to \mathscr{E}$ functors and $\mu\colon F \Rightarrow F', \eta\colon G \Rightarrow G'$ natural transformation. We then define,

$$\eta * \mu\colon GF \Rightarrow G'F'$$

where for each $X \in \mathscr{C}$,

$$\begin{aligned}(\eta * \mu)_X &= G'(\mu_X) \circ \eta_{F(X)} \\ &= \eta_{F'(X)} \circ G(\mu_X).\end{aligned}$$

The last equality is true by the naturality condition of η on the morphism μ_X in $\mathscr{B}$. This definition is equivalent to the composition of the following commutative diagram.

$$\begin{array}{ccc} GF(X) & \xrightarrow{\eta_{F(X)}} & G'F(X) \\ {\scriptstyle G(\mu_X)}\downarrow & \searrow^{(\eta*\mu)_X} & \downarrow{\scriptstyle G'(\mu_X)} \\ GF'(X) & \xrightarrow[\eta_{F'(X)}]{} & G'F'(X) \end{array}$$

Lemma 4.12. $\eta * \mu$ as defined above is a natural transformation.

Proof. Again we need to show the naturality condition holds for $\eta * \mu$. Given $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $f \in \hom(X, Y)$ we have,

$$\begin{aligned}(\eta * \mu)_Y \circ GF(f) &= \eta_{F'(Y)} \circ G'(\mu_Y) \circ GF(f) \\ &= \eta_{F'(Y)} \circ GF'(f) \circ G'(\mu_X) \\ &= G'F'(f) \circ \eta_{F'(X)} \circ G'(\mu_X) \\ &= G'F'(f) \circ (\eta * \mu)_X.\end{aligned}$$

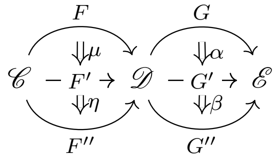
The third equality is true by the naturality of η and the second is true by the naturality of μ and that functors preserve commutativity (This is not proved here but can be found in Riehl [9]). This proof is equivalent to saying the diagram,

$$\begin{array}{ccccc} GF(X) & \xrightarrow{G'(\mu_X)} & GF'(X) & \xrightarrow{\eta_{F'(X)}} & G'F'(X) \\ {\scriptstyle GF(f)}\downarrow & & \downarrow{\scriptstyle GF'(f)} & & \downarrow{\scriptstyle G'F'(f)} \\ GF(Y) & \xrightarrow[G'(\mu_Y)]{} & GF'(Y) & \xrightarrow[\eta_{F'(Y)}]{} & G'F'(Y) \end{array}$$

commutes. □

4.2.3 Composition interchange

This section is adapted from Leinster [5] (Page 38). If we have the categories, functors and natural transformations as in the diagram below we can compose vertically and then horizontally or horizontally then vertically. We find this to be equivalent.



Theorem 4.13 (Composition interchange)**.** Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories, $\mathrm{F}, \mathrm{F}', \mathrm{F}'' \colon C \to D$ and $\mathrm{G}, \mathrm{G}', \mathrm{G}'' \colon D \to E$ functors and

$$\mu\colon F \Rightarrow F', \eta\colon F' \Rightarrow F'', \alpha\colon G \Rightarrow G', \beta\colon G' \Rightarrow G''$$

natural transformations. Then,

$$(\beta \cdot \alpha) * (\eta \cdot \mu) = (\beta * \eta) \cdot (\alpha * \mu). \tag{4.2}$$

Proof. For each $X \in \mathrm{Ob}(\mathscr{C})$ we have,

$$\begin{aligned}((\beta \cdot \alpha) * (\eta \cdot \mu))_X &= G''((\eta \cdot \mu)_X) \circ (\beta \cdot \alpha)_{F(X)} \\ &= G''(\eta_X \circ \mu_X) \circ \beta_{F(X)} \circ \alpha_{F(X)} \\ &= G''(\eta_X) \circ G''(\mu_X) \circ \beta_{F(X)} \circ \alpha_{F(X)} \\ &= G''(\eta_X) \circ (\beta_{F'(X)} \circ G'(\mu_X)) \circ \alpha_{F(X)} \\ &= (G''(\eta_X) \circ \beta_{F'(X)}) \circ (G'(\mu_X) \circ \alpha_{F(X)}) \\ &= (\beta * \eta)_X \circ (\alpha * \mu)_X \\ &= ((\beta * \eta) \cdot (\alpha * \mu))_X.\end{aligned}$$

Where the fourth equality is from the naturality of β. □

Remark 4.14. Equation (4.2) defines a natural transformation from GF to $G''F''$.

4.3 Functor categories

We can consider functors as objects and natural transformations as morphisms between them. Since we have a notion of composing natural transformations we just need to show there exists an identity transformation for each functor and that composition is associative and we will have a category.

Definition 4.15. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and F: $C \to D$ be a functor. Then define,

$$\begin{aligned}id_F \colon F &\Rightarrow F, \\ id_{F_X} &\mapsto id_{F(X)}.\end{aligned}$$

Lemma 4.16. id_F as defined above is a natural transformation.

Proof. We need to show the naturality condition holds. Given $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $f \in \hom(X, Y)$ we have,

$$\begin{aligned}id_{F_Y} \circ F(f) &= id_{F(Y)} \circ F(f) \\ &= F(f) \\ &= F(f) \circ id_{F(X)} \\ &= F(f) \circ id_{F_X}.\end{aligned}$$

□

Definition 4.17. Let $\mathscr{C}$,$\mathscr{D}$ be categories, F, G: $\mathscr{C} \to \mathscr{D}$ be functors. We call a natural transformation $\eta \colon F \Rightarrow G$ a *natural isomorphism* if there exists a natural transformation $\eta^{-1} \colon G \Rightarrow F$ such that,

$$\eta \cdot \eta^{-1} = id_G$$

and,

$$\eta^{-1} \cdot \eta = id_F.$$

Lemma 4.18. Let $\mathscr{C}$ and $\mathscr{D}$ be categories then for each functor F: $C \to D$ The natural transformation id_F acts as an identity with respect to vertical composition.

Proof. Given any functors from $\mathscr{C}$ to $\mathscr{D}$, $\mathrm{G}\colon C \to D$ and $\mathrm{G}'\colon C \to D$ and any two natural transformations $\eta\colon F \Rightarrow G$ and $\mu\colon G' \Rightarrow F$. Then for each $X \in \mathrm{Ob}(\mathscr{C})$,

$$\begin{aligned}(id_F \cdot \mu)_X &= id_{F_X} \circ \mu_X \\ &= id_{F(X)} \circ \mu_X \\ &= \mu_X\end{aligned}$$

and,

$$\begin{aligned}(\eta \cdot id_F)_X &= \eta_X \circ id_{F_X} \\ &= \eta_X \circ id_{F(X)} \\ &= \eta_X.\end{aligned}$$

□

Lemma 4.19. Vertical composition as defined in Definition 4.6 is associative.

Proof. Let $\mathrm{F}\colon C \to D$, $\mathrm{F}'\colon C \to D$, $\mathrm{G}'\colon C \to D$ and $\mathrm{G}\colon C \to D$ be functors. Let $\eta\colon F \Rightarrow F'$, $\mu\colon F' \Rightarrow G$ and $\gamma\colon G \Rightarrow G'$ be natural transformations. Then for each $X \in \mathrm{Ob}(C)$,

$$\begin{aligned}(\mu \cdot (\eta \cdot \gamma))_X &= \mu_X \circ (\eta_X \circ \gamma_X) \\ &= (\mu_X \circ \eta_X) \circ \gamma_X \\ &= ((\mu \cdot \eta) \cdot \gamma)_X\end{aligned}$$

where $\circ$ is associative since $\mathscr{C}$ is a category. □

Using Definition 2.1 we cannot state that for all categories $\mathscr{C}$ and $\mathscr{D}$ there exists a functor category with functors as the objects and natural transformations as the morphisms since it is possible to have class of natural transformations between two functors rather than a set we need to put a restriction on the size of the categories to guarantee the natural transformations between any two functors form a set. A similar theorem and proof of Theorem 4.23 is given in Riehl [9] (page 44 corollary 1.7.2) where here we specify small categories since Riehl uses a slightly different definition of a category.

Definition 4.20. A category $\mathscr{C}$ is called *small* if the collection of all morphisms in $\mathscr{C}$ form a set.

Remark 4.21. Definition 4.20 implies that the collection of all objects also forms a set since the objects are in a bijective correspondence with the set off all identity morphisms and these form a subset of the set of all morphisms in a category.

Definition 4.22. Let $\mathscr{C}$ and $\mathscr{D}$ be small categories. We define $\mathscr{D}^{\mathscr{C}} = (\mathrm{Ob}(\mathscr{D}^{\mathscr{C}}), \mathrm{hom}, \circ, id)$ as:

1. $\mathrm{Ob}(\mathscr{D}^{\mathscr{C}})$ is the collection of all functors between $\mathscr{C}$ and $\mathscr{D}$;
2. For each $F, G \in \mathrm{Ob}(\mathscr{D}^{\mathscr{C}})$, $\mathrm{hom}_{\mathscr{D}^{\mathscr{C}}}(F, G)$ is the collection of natural transformations between F and G;
3. $\circ$ is vertical composition of natural transformations as in Definition 4.6;
4. For each $F \in \mathrm{Ob}(\mathscr{D}^{\mathscr{C}})$ we have the identity id_F defined in Definition 4.15.

Theorem 4.23. $\mathscr{D}^{\mathscr{C}}$ defined above in Definition 4.22 is a small category.

Proof. First note that the collection of all functors between $\mathscr{C}$ and $\mathscr{D}$ is indeed a set since the collection of all functions between two sets forms a set. Similarly, the collection of all natural transformations between two functors is a set. Composition is defined in Definition 4.6 which is associative by Lemma 4.19. Identities exist and are defined by Definition 4.15. □

The following example was stated on Wiki [12] where here we go in to more detail.

Example 4.24. Recall the category of sets, **Set** as defined in Example 2.4 and consider a monoid $\mathscr{M}$ defined as a one object category as in Subsection 2.1.2. The functor category $\mathbf{Set}^{\mathscr{M}}$ is the category whose objects are the left action functors, L_X, as defined in Example 3.23 and morphisms are the natural transformations defined in Example 4.5. This category represents the category of sets acted on by left action of the monoid $\mathscr{M}$. The category $\mathbf{Set}^{\mathscr{M}}$ is isomorphic to a wide subcategory of the category **Set**. Let the functor $\sigma\colon \mathbf{Set}^{\mathscr{M}} \to \mathbf{Set}$ be defined on objects,

$$L_X \mapsto X$$

and for a morphism $\alpha \in \hom_{\mathbf{Set}^{\mathscr{M}}}(L_X, L_Y)$,

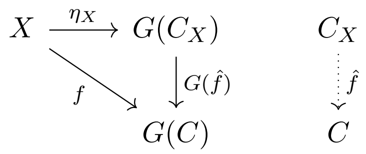
$$\alpha \mapsto \alpha_m$$

as in defined in Example 4.5. Then σ is an isomorphism from $\mathbf{Set}^{\mathscr{M}}$ to the subcategory of **Set** which has for morphisms only the functions of the form α_M.

5 Universal morphisms and adjoint functors

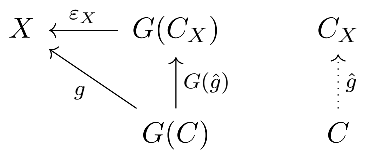
In this section we see how the free and forgetful functors defined in Definition 3.4 and Definition 3.6 are related by looking at universal morphism and adjoint functors. Definitions and notation from this section are from Clementino [3] and Adámek - Herrlick - Strecker [1].

Definition 5.1 (Universal morphism)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $\mathrm{G}\colon C \to D$ be a functors and $X \in \mathrm{Ob}(\mathscr{D})$.
A *universal morphism from* X *to* G is a pair (η_X, C_X) where $\eta_X \in \hom(X, G(C_X))$ is a morphism and $C_X \in \mathrm{Ob}(\mathscr{C})$ such that for each $C \in \mathrm{Ob}(\mathscr{C})$ and each morphism $f \in \hom(X, G(C))$ there exists a unique morphism $\hat{f} \in \hom(C_X, C)$ for which the diagram on the left



(5.1)

commutes.
A *universal morphism from* G *to* X is a pair (ε_X, C_X) where $\varepsilon_X \in \hom(G(C_X), X)$ is a morphism and $C_X \in \mathrm{Ob}(\mathscr{C})$ such that for each $C \in \mathrm{Ob}(\mathscr{C})$ and each morphism $g \in \hom(G(C), X)$ there exists a unique morphism $\hat{g} \in \hom(C, C_X)$ for which the diagram on the left



commutes.

Definition 5.2 (Adjoint)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $\mathrm{G}\colon \mathscr{C} \to \mathscr{D}$ a functor. We say G is *right adjoint* if for each object $X \in \mathrm{Ob}(\mathscr{D})$ there exists a universal morphism, (η_X, C_X) from X to G. We say G is *left adjoint* if for each object $X \in \mathrm{Ob}(\mathscr{D})$ there exists a universal morphism, (ϵ_X, C_X) from G to X. We say G is *adjoint* if G is either right adjoint or left adjoint.

5.1 Example: Free/forgetful adjunction for monoids

Lemma 5.3. Let $\mathrm{F}\colon \mathbf{Set} \to \mathbf{Mon}$ be the free monoid functor defined in Definition 3.9 and $\mathrm{U}\colon \mathbf{Mon} \to \mathbf{Set}$ be the forgetful monoid functor as defined in Definition 3.4. Then F is left adjoint.

Proof. For each $X \in \mathrm{Ob}(\mathbf{Set})$ let

$$\begin{aligned} \eta\colon id_{\mathbf{Set}} &\to UF, \\ \eta_X\colon X &\to U((\gamma(X), *, \varnothing)), \\ x &\mapsto (x). \end{aligned}$$

Then given $(A, \circ, e_A) \in \mathbf{Mon}$ and $f \in \hom_{\mathbf{Set}}(X, U((A, \circ, e_A)))$ let

$$\begin{aligned} \hat{f}\colon (\gamma(X), *, \varnothing) &\to (A, \circ, e_A), \\ \hat{f}((x_1, x_2, \dots, x_n)) &\mapsto f(x_1) \circ f(x_2) \circ \dots \circ f(x_n), \\ \hat{f}(\varnothing) &\mapsto e_A. \end{aligned}$$

We first show $\hat{f}$ is a monoid homomorphism. We ,have that

$$\hat{f}(\varnothing) = e_A$$

and,

$$\begin{aligned}\hat{f}((x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)) &= f(x_1) \circ f(x_2) \circ \cdots \circ f(x_n) \circ f(y_1) \circ f(y_2) \circ \cdots \circ f(y_n)) \\ &= \hat{f}((x_1, x_2, \ldots, x_n)) \circ \hat{f}((y_1, y_2, \ldots, y_n)).\end{aligned}$$

Hence $\hat{f}$ is a monoid homomorphism. The diagram,

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & U((\gamma(X), *, \varnothing)) & & (\gamma(X), *, \varnothing) \\ & \searrow^{f} & \downarrow{\scriptstyle U(\hat{f})} & & \vdots\downarrow{\scriptstyle \hat{f}} \\ & & U((A, \circ, e_A)) & & (A, \circ, e_A) \end{array}$$

Commutes. So we need to show $\hat{f}$ is unique. Assume there exists monoid homomorphism $\hat{g}$ such that the diagram commutes. Then for each $x \in X$,

$$\begin{aligned} f(x) &= \hat{g}(\eta_x(x)) \\ &= \hat{g}((x)) \end{aligned}$$

Since $\hat{g}$ is a monoid homomorphism,

$$\begin{aligned}\hat{g}((x_1, x_2, \ldots, x_n)) &= \hat{g}((x_1)) \circ \hat{g}((x_2)) \cdots \circ \hat{g}((x_n)) \\ &= f(x_1) \circ f(x_2) \circ \cdots \circ f(x_n) \\ &= \hat{f}.\end{aligned}$$

Therefore, since there is a unique $\hat{f}$ such that the diagram commutes for all $X \in \mathbf{Set}$. F is left adjoint. □

Remark 5.4. We will see later that Lemma 5.3 implies U is right adjoint.

5.1.1 Equivalent definitions of adjoint functors

The following Lemma 5.5 followed from discussion with the supervisor and from the Wikipedia article on adjoint functors [10]. Here we add the proof.

Lemma 5.5. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, $\mathrm{G}\colon \mathscr{C} \to \mathscr{D}$ be a functor and for each $X \in \mathrm{Ob}(\mathscr{D})$ we have (η_X, C_X) is a universal morphism from X to G. Then given $C \in \mathscr{C}$ there exists a bijective function, defined as

$$\begin{aligned}\phi_{X,C}\colon \hom_{\mathscr{D}}(X, G(C)) &\to \hom_{\mathscr{C}}(C_X, C), \\ f &\mapsto \hat{f},\end{aligned}$$

where $\hat{f}$ is defined from the universal property in Diagram (5.1). Further given $C' \in \mathrm{Ob}(\mathscr{C})$ and $g \in \hom(C, C')$ the following diagram commutes,

$$\begin{array}{ccc} \hom_{\mathscr{D}}(X, G(C)) & \xrightarrow{\phi_{X,C}} & \hom_{\mathscr{C}}(C_X, C) \\ {\scriptstyle \mathrm{h}_X(G(g))}\downarrow & & \downarrow{\scriptstyle \mathrm{h}_{C_X}(g)} \\ \hom_{\mathscr{D}}(X, G(C')) & \xrightarrow[\phi_{X,C'}]{} & \hom_{\mathscr{C}}(C_X, C') \end{array} \tag{5.2}$$

where h_{C_X} is the functor defined in Lemma 3.12. In particular the $\phi_{X,C}$, where $C \in \mathrm{Ob}(\mathscr{C})$, combine to give a natural isomorphism $\phi_X \colon \mathrm{h}_X \circ G \Rightarrow \mathrm{h}_{C_X}$.

Proof. First we show that for $C \in \mathrm{Ob}(\mathscr{C})$, $\phi_{X,C}$ is a bijection by showing it is surjective and injective. Given $f, f' \in \hom_{\mathscr{D}}(X, G(C))$ suppose $\hat{f} = \hat{f}'$, then we have,

$$\begin{aligned} f &= G(\hat{f}) \circ \eta_X \\ &= G(\hat{f}') \circ \eta_X \\ &= f'. \end{aligned}$$

Hence $\phi_{X,C}$ is injective. Given $g \in \hom_{\mathscr{C}}(C_X, C)$ we have $G(g) \in \hom_{\mathscr{D}}(G(C_X), G(C))$ since G is a functor. We can then construct $f = G(g) \circ \eta_X$ since $\eta_X \in \hom_{\mathscr{D}}(X, G(C_X))$ and $\mathscr{D}$ is a category, hence $g = \hat{f}$. Therefore $\phi_{X,C}$ is surjective and hence a bijection. We will now show the diagram (5.2) commutes. Given $f \in \hom_{\mathscr{D}}(X, G(C))$ we need to show

$$g \circ \phi_{X,C}(f) = \phi_{X,C'}(G(g) \circ f).$$

We have,

$$\begin{aligned} g \circ \phi_{X,C}(f) &= g \circ \hat{f} \\ &= \phi_{X,C'}(\phi^{-1}_{X,C'}(g \circ \hat{f})) \\ &= \phi_{X,C'}(G(g \circ \hat{f}) \circ \eta_X) \\ &= \phi_{X,C'}(G(g) \circ G(\hat{f}) \circ \eta_X) \\ &= \phi_{X,C'}(G(g) \circ f). \end{aligned}$$

□

The following theorem is adapted from Adámek - Herrlick - Strecker [1] (page 306 Theorem 19.1) where here we add the complete proof which was left as an exercise.

Theorem 5.6. Let $\mathrm{G} \colon \mathscr{C} \to \mathscr{D}$ be a right adjoint functor and suppose that for each object $X \in \mathrm{Ob}(\mathscr{C})$ we are given a universal morphism (η_X, C_X), from X to G.

1. There exists a unique functor $\mathrm{F} \colon \mathscr{D} \to \mathscr{C}$ such that the following two conditions hold:
   (a) $F(X) = C_X$;
   (b) We have a natural transformation,
   $$\eta \colon id_{\mathscr{D}} \Rightarrow GF$$
   whose components are given by:
   $$\eta_X \colon X \to G(C_X);$$
2. Further, we have a natural transformation $\varepsilon \colon FG \Rightarrow id_{\mathscr{C}}$ where for each $C \in \mathscr{C}$, ε_C is the unique morphism for which,
$$\begin{array}{ccc} G(C) & \xrightarrow{\eta_{G(C)}} & G(C_X) \\ & \searrow^{id_{G(C)}} & \downarrow G(\varepsilon_C) \\ & & G(C) \end{array} \tag{5.3}$$
commutes;

3. We also have that the following identities are satisfied:
   (a) $\eta G \circ G\varepsilon = id_G$;
   (b) $F\eta \circ \varepsilon F = id_F$.

Proof. Let F: $\mathscr{D} \to \mathscr{C}$ be defined:

$$\begin{aligned} F\colon \mathrm{Ob}(\mathscr{D}) &\to \mathrm{Ob}(\mathscr{C}), \\ X &\mapsto C_X \end{aligned}$$

and,

$$F\colon \ \hom_{\mathscr{D}}(X,Y) \to \hom_{\mathscr{C}}(F(X),F(Y)),\ f \qquad \mapsto \widehat{\eta_Y \circ f}.$$

This definition comes from the diagram,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(C_X) \\ {\scriptstyle f}\downarrow & \overset{\eta_Y \circ f}{\searrow} & \downarrow{\scriptstyle G(\widehat{\eta_Y \circ f})} \\ Y & \xrightarrow[\eta_Y]{} & G(C_Y) \end{array}$$

which commutes since η_X is a universal morphism. If this is a functor then it is unique since it is unique on objects and for $f \in \hom_{\mathscr{D}}(X,Y)$ we have $\eta_Y \circ f \in \hom_{\mathscr{D}}(X, G(C_Y))$ and by Lemma 5.5 there is a bijection between $\hom_{\mathscr{D}}(X, G(C_Y))$ and $\hom_{\mathscr{C}}(C_X, C_Y)$ hence F is unique on morphisms.

We show F is a functor. Given $X \in \mathrm{Ob}(\mathscr{D})$,

$$\begin{aligned} F(id_X) &= \widehat{\eta_X \circ id_X} \\ &= \hat{\eta_X} \\ &= id_{C_X} \\ &= id_{F(X)}. \end{aligned}$$

Equivalently the diagram,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(C_X) \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle G(\widehat{\eta_X \circ id_X})} \\ X & \xrightarrow[\eta_X]{} & G(C_X) \end{array}$$

commutes since η_X is a universal morphism. We also have for $f \in \hom_{\mathscr{D}}(X,Y)$ and $g \in \hom_{\mathscr{D}}(Y,Z)$, the diagram,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(C_X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(\widehat{\eta_Y \circ f})} \\ Y & \xrightarrow[\eta_Y]{} & G(C_Y) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G(\widehat{\eta_Z \circ g})} \\ Z & \xrightarrow[\eta_Z]{} & G(C_Z) \end{array}$$

commutes since η_X and η_Y are universal morphisms and we also have,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(C_X) \\ {\scriptstyle g\circ f}\downarrow & & \downarrow{\scriptstyle G(\eta_Z\widehat{\circ(g\circ f}))} \\ Z & \xrightarrow[\eta_Z]{} & G(C_Z) \end{array}$$

commutes since η_X is a universal morphism, in particular,

$$G(\eta_Z \widehat{\circ (g \circ} f) = G(\widehat{\eta_Z \circ g}) \circ G(\widehat{\eta_Y \circ f}),$$

therefore, since G is a functor,

$$\begin{aligned} F(g \circ f) &= \eta_Z \widehat{\circ g \circ} f \\ &= \widehat{\eta_Z \circ g} \circ \widehat{\eta_Y \circ f} \\ &= F(g) \circ F(f). \end{aligned}$$

Hence F is a functor.

Define $\eta\colon id_{\mathscr{D}} \Rightarrow FG$ for each $X \in \mathrm{Ob}(\mathscr{D})$, η_X is the universal morphism given. Then clearly $\eta_X \in \hom_{\mathscr{D}}(X, GF(X))$ since $F(X) = C_X$.

We show the naturality condition holds. Given $X, Y \in \mathrm{Ob}(\mathscr{D})$ and $f \in \hom_{\mathscr{D}}(X, Y)$,

$$\begin{array}{ccc} id_{\mathscr{D}}(X) & \xrightarrow{\eta_X} & GF(X) \\ {\scriptstyle id_{\mathscr{D}}(f)}\downarrow & & \downarrow{\scriptstyle GF(f)} \\ id_{\mathscr{D}}(Y) & \xrightarrow[\eta_Y]{} & GF(Y) \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{\eta_X} & G(C_X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(\widehat{\eta_Y \circ f})} \\ Y & \xrightarrow[\eta_Y]{} & G(C_Y) \end{array}$$

commutes since η_X is a universal morphism. Hence the naturality condition holds and η is a natural transformation. $\varepsilon\colon FG \Rightarrow id_{\mathscr{C}}$ exists since $\eta_{G(C)}$ is a universal morphism we show the naturality condition holds. Given $C, C' \in \mathrm{Ob}(C)$ and $f \in \hom_{\mathscr{C}}(C, C')$ we have,

$$\begin{aligned} G(f \circ \varepsilon_C) \circ \eta_{G(C)} &= G(f) \circ G(\varepsilon_C) \circ \eta_{G(C)} \\ &= G(f) \circ id_{G(C)} \\ &= G(f) \\ &= G(\varepsilon_{C'}) \circ \eta_{G(C')} G(f) \\ &= G(\varepsilon_{C'}) \circ GFG(f) \circ \eta_{G(C)}, && \text{by the naturality of } \eta, \\ &= G(\varepsilon_{C'} \circ FG(f)) \circ \eta_{G(C)}. \end{aligned}$$

Therefore,

$$f \circ \varepsilon_C = \varepsilon_{C'} \circ FG(f).$$

The identity (a) is satisfied by the definition of ε Diagram (5.3). To show identity (b) first note since η is a natural transformation from $id_{\mathscr{D}}$ to GF we have for each $D \in \mathrm{Ob}(\mathscr{D})$,

$$\begin{aligned} (\eta GF \circ \eta)_D &= \eta_{G(F(D))} \circ \eta_D \\ &= G(F(\eta_D)) \circ \eta_D \\ &= (GF\eta \circ \eta)_D. \end{aligned}$$

Therefore by using identity (a) we have:

$$\begin{aligned} G(id_F) \circ \eta &= id_G F \circ \eta \\ &= G\varepsilon F \circ \eta GF \circ \eta \\ &= G\varepsilon F \circ GF\eta \circ \eta \\ &= G(\varepsilon F \circ F\eta) \circ \eta. \end{aligned}$$

Hence,

$$id_F = \varepsilon F \circ F\eta.$$

□

5.2 Adjoint situations

Definition 5.7 if adapted from Adámek - Herrlick - Strecker [1] page 307.

Definition 5.7. An *adjoint situation* $(F, G, \eta, \varepsilon)$ is a pair of functors $\mathrm{F}\colon \mathscr{D} \to \mathscr{C}$ and $\mathrm{G}\colon \mathscr{C} \to \mathscr{D}$ and a pair of natural transformations $\eta\colon id_{\mathscr{D}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow id_{\mathscr{C}}$. Satisfying the *triangle identities*:

1. $\varepsilon F \circ F\eta = id_F$;
2. $G\varepsilon \circ \eta G = id_G$.

We call η the *unit* and ε the *co-unit*.

Remark 5.8. Given a right adjoint functor $\mathrm{G}\colon \mathscr{C} \to \mathscr{D}$, Theorem 5.6 tells us that there exists at least one adjoint situation $(F, G, \eta, \varepsilon)$.

The following Corollary 5.9 followed from discussions with the supervisor and here we provide a proof.

Corollary 5.9. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, $\mathrm{G}\colon \mathscr{C} \to \mathscr{D}$ be a functor. The following are equivalent:

1. For each $D \in \mathrm{Ob}(\mathscr{D})$, $(\eta_D, F(D))$is a universal morphism from D to G;
2. $(F, G, \eta, \varepsilon)$ is an adjoint situation;
3. The family of functions:

   $$\phi_{D,C}\colon\ \hom_{\mathscr{C}}(F(D), C) \to \hom_{\mathscr{D}}(D, G(C)),$$

   are bijections. That is for all $D \in \mathrm{Ob}(\mathscr{D})$, there exists a natural isomorphism $\phi_D\colon \mathrm{h}_D \circ G \Rightarrow \mathrm{h}_{G(C)}$.

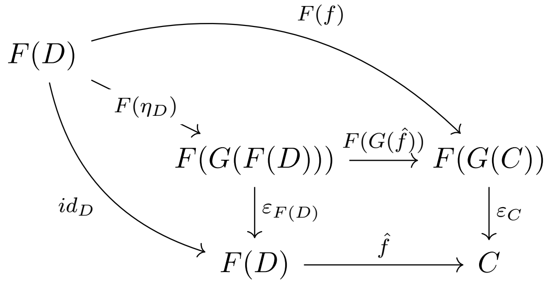
Proof. (1) $\implies$ (2) by Theorem 5.6 and (1) $\implies$ (3) by Lemma 5.5. Suppose that for each object $D \in \mathrm{Ob}(\mathscr{D})$ we have a natural isomorphism $\phi_D\colon \mathrm{h}_D \circ G \Rightarrow \mathrm{h}_{G(C)}$. Then for each object $D \in \mathrm{Ob}(\mathscr{D})$ and each $C \in \mathrm{Ob}(\mathscr{C})$, $f \in \hom_{\mathscr{D}}(D, G(C))$ there is a unique morphism $\phi_D(f) \in \hom_{\mathscr{C}}(F(D), C)$, therefore $(\eta_D, F(D))$ where η_D is the unique morphism for which the diagram,

$$\begin{array}{ccccc} D & \xrightarrow{\eta_D} & G(F(D)) & \qquad & F(D) \\ & \searrow^{f} & \big\downarrow{\scriptstyle G(\phi_D(f))} & & \vdots\,{\scriptstyle \phi_D(f)} \\ & & G(C) & & C \end{array}$$

commutes. Therefore (3) $\implies$ (1). Suppose we have an adjoint situation $(F, G, \eta, \varepsilon)$ then for $C \in \mathrm{Ob}(\mathscr{C})$, $D \in \mathrm{Ob}(\mathscr{D})$ and $f \in \hom_{\mathscr{D}}(D, G(C))$ the diagram on the left,

$$\begin{array}{ccccc} D & \xrightarrow{\eta_D} & G(F(D)) & & \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle G(F(f))} & & \\ G(C) & \xrightarrow{\eta_{G(C)}} & G(F(G(C))) & \qquad & F(D) \\ & \searrow^{id_{G(C)}} & \big\downarrow{\scriptstyle G(\varepsilon_C)} & & \vdots\,{\scriptstyle \varepsilon_C \circ F(f)} \\ & & G(C) & & C \end{array}$$

commutes. Therefore for a morphism f there exists a morphism $\hat{f} = \varepsilon_C \circ F(f)$ for which the diagram commutes. To show uniqueness if $\hat{f} \in \hom_{\mathscr{C}}(F(D), C)$ where $f = G(\hat{f}) \circ \eta_D$ we have the diagram,



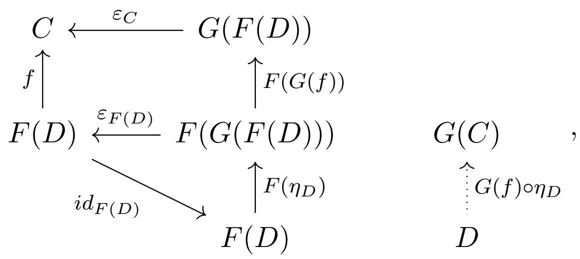
commutes, and therefore $\hat{f} = \varepsilon_C \circ F(f)$ is the unique morphism for which $f = G(\hat{f}) \circ \eta_D$, hence $(\eta_D, F(D))$ is a universal morphism from D to G for all $D \in \mathrm{Ob}(\mathscr{D})$. Therefore (2) $\implies$ (1). □

The following Lemma 5.10 is adapted from Adámek - Herrlick - Strecker [1] (Page 308 Proposition 19.7)

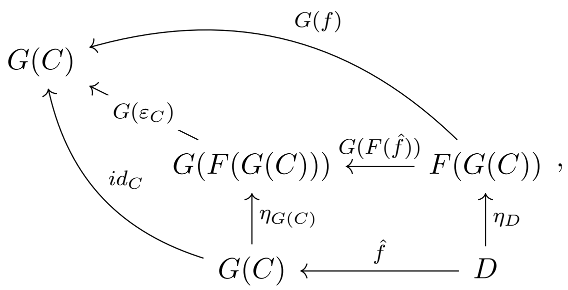
Lemma 5.10. Given an adjoint situation $(F, G, \eta, \varepsilon)$ we have:

1. G is a right adjoint functor.
2. For each $D \in \mathrm{Ob}(\mathscr{D})$, $(\eta_D, F(D))$ is a universal morphism from D to G.
3. F is a left adjoint functor.
4. For each $C \in \mathrm{Ob}(\mathscr{C})$, $(\varepsilon_C, G(C))$ is a universal morphism from F to C

Proof. By Corollary 5.9 if we have an adjoint situation, $(F, G, \eta, \varepsilon)$ then $(\eta_D, F(D))$ are universal morphisms hence G is right adjoint. The proof for (4) and thus (3) follows similarly to the proof of 5.9. Suppose we have the adjoint situation $(F, G, \eta, \varepsilon)$, then for $D \in \mathrm{Ob}(\mathscr{D})$, $C \in \mathrm{Ob}(\mathscr{C})$ and $f \in \hom_{\mathscr{C}}(F(D), C)$ the diagram on the left,



commutes. Therefore for a morphism f there exists a morphism $\hat{f} = G(f) \circ \eta_D$ for which the diagram commutes. To show uniqueness suppose we have $\hat{f} \in \hom_{\mathscr{D}}(G(C), D)$ where $f = \varepsilon_C \circ F(\hat{f})$, then we have the diagram,



commutes, and therefore $\hat{f} = G(f) \circ \eta_D$ is the unique morphism for which $f = \varepsilon_C \circ F(\hat{f})$, hence $(\varepsilon_C, G(C))$ is a universal morphism from F to C for all $C \in \mathrm{Ob}(\mathscr{D})$ and therefore F is a left adjoint functor. □

The following Lemma 5.11 is adapted from Adámek - Herrlick - Strecker [1] (Page 309 Proposition 19.9) however here we give the proof from the perspective of the left adjoint functor F.

Lemma 5.11. Let F: $\mathscr{D} \to \mathscr{C}$ be a left adjoint functor and $(F, G, \eta, \varepsilon)$ be an adjoint situation.

1. If there is an adjoint situation $(F, G', \eta', \varepsilon')$ then there exists a natural isomorphism $\tau\colon G \Rightarrow G'$ where $\varepsilon' = F\tau \circ \varepsilon$ and $\eta' = \eta \circ \tau^{-1}F$;

2. If we have a functor G' and a natural isomorphism $\tau\colon G \Rightarrow G'$ then $(F, G', \eta \circ \tau^{-1}F, F\tau \circ \varepsilon)$ is an adjoint situation.

Proof. (1): By Lemma 5.10 we have for each $C \in \mathrm{Ob}(\mathscr{C})$, $(\varepsilon_C, G(C))$ and $(\varepsilon'_C, G'(C))$ are universal morphisms from F to G. Therefore there is an isomorphism τ_C with $\varepsilon'_C = F\tau_C \circ \varepsilon_C$ by Definition 5.1 and hence $\tau\colon G \Rightarrow G'$ is a natural isomorphism with $\varepsilon' = F\tau \circ \varepsilon$. For each $D \in \mathrm{Ob}(\mathscr{D})$ we have:

$$\begin{aligned}
\varepsilon_{F(D)} \circ F(\eta_D) &= id_{F(D)} \\
&= \varepsilon'_{F(D)} \circ F(\eta'_D) \\
&= F\tau_{F(D)} \circ \varepsilon_{F(D)} \circ F(\eta'_D) \\
&= \varepsilon_{F(D)} \circ F(\tau_{F(D)} \circ \eta'_D).
\end{aligned}$$

Therefore $\eta_D = \tau_{F(D)} \circ \eta'_D$, and hence $\eta' = \eta \circ \tau^{-1}F$.

(2): We have for each $D \in \mathrm{Ob}(\mathscr{D})$,

$$\begin{aligned}
(F(\eta \circ \tau^{-1}F) \circ (F\tau \circ \varepsilon)F)(D) &= F(\eta_D) \circ F(\tau^{-1}_{F(D)}) \circ F(\tau_{F(D)}) \circ \varepsilon_{F(D)} \\
&= F(\eta_D) \circ F(\tau^{-1}_{F(D)} \circ \tau_{F(D)}) \circ \varepsilon_{F(D)} \\
&= F(\eta_D) \circ \varepsilon_{F(D)} \\
&= id_F(D)
\end{aligned}$$

and for $C \in \mathrm{Ob}(\mathscr{C})$ we have,

$$\begin{aligned}
(\eta \circ \tau^{-1}F)G' \circ G'(F\tau \circ \varepsilon)(C) &= \eta_{G'(C)} \circ \tau^{-1}_{F(G'(C))} \circ G'(F(\tau_C) \circ \varepsilon_C) \\
&= \eta_{G'(C)} \circ \tau^{-1}_{F(G'(C))} \circ G'(F(\tau_C)) \circ G'(\varepsilon_C) \\
&= \eta_{G'(C)} \circ G'(\varepsilon_C) \\
&= id_{G'}(C).
\end{aligned}$$

Therefore $(F, G', \eta \circ \tau^{-1}F, F\tau \circ \varepsilon)$ is an adjoint situation by Definition 5.7. □

5.2.1 Category of adjoint situations

This section uses ideas from MacLane [6], specifically chapter IV. To define a category of adjoint situations we need to define morphisms between adjoint situations. The following Definition 5.12 is adapted from MacLane [6] (Page 99).

Definition 5.12. Let $(\mathrm{F}\colon \mathscr{D} \to \mathscr{C}, \mathrm{G}\colon \mathscr{C} \to \mathscr{D}, \eta, \varepsilon)$ and $(\mathrm{F}'\colon \mathscr{D}' \to \mathscr{C}', \mathrm{G}'\colon \mathscr{C}' \to \mathscr{D}', \eta', \varepsilon')$ be adjoint situations. A *morphism between adjoint situations* from $(F, G, \eta, \varepsilon)$ to $(F', G', \eta', \varepsilon')$ is a pair of functors $(\mathrm{K}\colon \mathscr{D} \to \mathscr{D}', \mathrm{L}\colon \mathscr{C} \to \mathscr{C}')$ such that:

1. For each object $C \in \mathrm{Ob}(\mathscr{C})$,

$$(K \circ F \circ G)(C) = (F' \circ G' \circ K)(C) = (F' \circ L \circ G)(C)$$

and each morphism $f \in \hom_{\mathscr{C}}(C, C')$

$$(K \circ F \circ G)(f) = (F' \circ G' \circ K)(f) = (F' \circ L \circ G)(f).$$

That is the diagram,

$$\begin{CD} \mathscr{C} @>G>> \mathscr{D} @>F>> \mathscr{C} \\ @VKVV @VLVV @VKVV \\ \mathscr{C}' @>G'>> \mathscr{D}' @>F'>> \mathscr{C}' \end{CD} \tag{5.4}$$

commutes;

2. For each object $C \in \mathrm{Ob}(\mathscr{C})$ we have,

$$\varepsilon'_{K(C)} = K(\varepsilon_C),$$

and for each object $D \in \mathrm{Ob}(\mathscr{D})$ we have,

$$L(\eta_D) = \eta'_{L(D)}.$$

Lemma 5.13. Let $(\mathrm{F}\colon \mathscr{D} \to \mathscr{C}, \mathrm{G}\colon \mathscr{C} \to \mathscr{D}, \eta, \varepsilon)$, $(\mathrm{F}'\colon \mathscr{D}' \to \mathscr{C}', \mathrm{G}'\colon \mathscr{C}' \to \mathscr{D}', \eta', \varepsilon')$ and $(\mathrm{F}''\colon \mathscr{D}'' \to \mathscr{C}'', \mathrm{G}''\colon \mathscr{C}'' \to \mathscr{D}'', \eta'', \varepsilon'')$ be adjoint situations and let $(\mathrm{K}\colon \mathscr{D} \to \mathscr{D}', \mathrm{L}\colon \mathscr{C} \to \mathscr{C}')$ and $(\mathrm{K}'\colon \mathscr{D}' \to \mathscr{D}'', \mathrm{L}'\colon \mathscr{C}' \to \mathscr{C}'')$ be morphisms of adjoint situations. Then $(K' \circ K, L' \circ L)$ is a morphism of adjoint situations.

Proof. First note $\mathrm{K}' \circ \mathrm{K}\colon \mathscr{D} \to \mathscr{D}'$ and $\mathrm{L}' \circ \mathrm{L}\colon \mathscr{C} \to \mathscr{C}'$ are functors since they are the composition of functors. We have the diagram,

$$\begin{CD} \mathscr{C} @>G>> \mathscr{D} @>F>> \mathscr{C} \\ @VKVV @VLVV @VKVV \\ \mathscr{C}' @>G'>> \mathscr{D}' @>F'>> \mathscr{C}' \\ @VK'VV @VL'VV @VK'VV \\ \mathscr{C}'' @>G''>> \mathscr{D}'' @>F''>> \mathscr{C}'' \end{CD}$$

commutes since all parts of the diagram commute. Then for each $C \in \mathrm{Ob}(\mathscr{C})$ we have the following,

$$\begin{aligned} \varepsilon''_{K'(K(C))} &= K'(\varepsilon'_{K(C)}) \\ &= K'(K(\varepsilon_C)) \end{aligned}$$

and for each $D \in \mathrm{Ob}(\mathscr{D})$ we have,

$$\begin{aligned} L'(L(\eta_D)) &= L'(\eta_{L(D)}) \\ &= \eta_{L'(L(D))}. \end{aligned}$$

Therefore $(K' \circ K, L' \circ L)$ is a morphism of adjoint situations. ☐

Definition 5.14. We define the composition of two morphisms of adjoint situations, (L, K) and (L', L') as

$$(L', K') \circ (L, K) = (K' \circ K, L' \circ L)$$

Example 5.15. Let $(\mathrm{F}\colon \mathscr{D} \to \mathscr{C}, \mathrm{G}\colon \mathscr{C} \to \mathscr{D}, \eta, \varepsilon)$ be an adjoint situation. Then $(id_{\mathscr{C}}, id_{\mathscr{D}})$ a morphism of adjoint situations from $(F, G, \eta, \varepsilon)$ to $(F, G, \eta, \varepsilon)$. In fact we have that this morphism acts as an identity with respect to morphisms of adjoint situations and the composition defined above in Definition 5.14.

Proof. We need to show Diagram 5.4 commutes, so we have,

$$\begin{aligned} id_{\mathscr{C}} \circ F \circ G &= F \circ G \\ &= F \circ G \circ id_{\mathscr{C}} \\ &= F \circ G \\ &= F \circ id_{\mathscr{D}} \circ G. \end{aligned}$$

We also have for each $C \in \mathrm{Ob}(\mathscr{C})$,

$$\begin{aligned} \varepsilon_{id_C(C)} &= \varepsilon_C \\ &= id_{\mathscr{C}}(\varepsilon_C) \end{aligned}$$

and each $D \in \mathrm{Ob}(\mathscr{D})$,

$$\begin{aligned} id_{\mathscr{D}}(\eta_D) &= \eta_D \\ &= \eta_{id_{\mathscr{C}}(D)}. \end{aligned}$$

$(id_{\mathscr{C}}, id_{\mathscr{D}})$ clearly acts as an identity since for any morphism (L, K) from any adjoint situation to $(F, G, \eta, \varepsilon)$ and (L', K') from $(F, G, \eta, \varepsilon)$ to any adjoint situation,

$$\begin{aligned} (L, K) \circ (id_{\mathscr{C}}, id_{\mathscr{D}}) &= (L \circ id_{\mathscr{C}}, K \circ id_{\mathscr{D}}) \\ &= (L, K) \end{aligned}$$

and,

$$\begin{aligned} (id_{\mathscr{C}}, id_{\mathscr{D}}) \circ (K', L') &= (id_{\mathscr{C}} \circ K', id_{\mathscr{D}} \circ L') \\ &= (L', K'). \end{aligned}$$

□

5.3 Composition of adjoint functors

The next Definition 5.16 is adapted from [1] (Page 309, Definition 19.10).

Definition 5.16. Let $\mathrm{G}\colon \mathscr{C} \to \mathscr{D}$ and $\mathrm{F}\colon \mathscr{D} \to \mathscr{C}$ be functors. F is *left adjoint to* G and G is *right adjoint to* F, written $F \dashv G$ if there exists an adjoint situation $(F, G, \eta, \varepsilon)$.

The following Lemma 5.17 is adapted from Leinster [5] (Page 49, Remark 2.1.11) here we add a proof.

Lemma 5.17. Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be categories, $\mathrm{G}\colon \mathscr{C} \to \mathscr{D}$ be right adjoint to $\mathrm{F}\colon \mathscr{D} \to \mathscr{C}$ and $\mathrm{G}'\colon \mathscr{D} \to \mathscr{E}$ be right adjoint to $\mathrm{F}'\colon \mathscr{E} \to \mathscr{D}$. Then $\mathrm{G}' \circ \mathrm{G}\colon \mathscr{C} \to \mathscr{E}$ is right adjoint to $\mathrm{F} \circ \mathrm{F}'\colon \mathscr{E} \to \mathscr{D}$ and given $C \in \mathrm{Ob}(\mathscr{C})$ and $E \in \mathrm{Ob}(\mathscr{E})$ we have,

$$\hom_{\mathscr{C}}(F(F'(E)), C) \cong \hom_{\mathscr{D}}(F'(E), G(C)) \cong \hom_{\mathscr{E}}(E, G'(G(C))).$$

Proof. We have for any $E \in \mathrm{Ob}(\mathscr{E})$, $F'(E) \in \mathrm{Ob}(\mathscr{D})$ hence since there is an adjoint situation F,G

$$\hom_{\mathscr{C}}(F(F'(E)), C) \cong \hom_{\mathscr{D}}(F'(E), G(C)).$$

Similarly, for any $C \in \mathrm{Ob}(\mathscr{C})$, $G(C) \in \mathrm{Ob}(\mathscr{D})$ hence since there is an adjoint situation G', F' we have,

$$\hom_{\mathscr{D}}(F'(E), G(C)) \cong \hom_{\mathscr{E}}(E, G'(G(C))).$$

□

Remark 5.18. Lemma 5.17 shows that if we compose two adjoint functors then we get an adjoint functor.

5.4 Examples

The following Example 5.19 came from a discussion with the supervisor and can be found in Leinster [5] (Page 47, Example 2.16). Example 5.19 can also be found in Adámek - Herrlick - Strecker [1] (Page 307, Example 19.4 (3)).

Example 5.19. Let $M \in \mathrm{Ob}(\mathbf{Set})$ be a set then we have a functor

$$(-) \times M \colon \mathbf{Set} \to \mathbf{Set}$$

defined for each object $X \in \mathrm{Ob}(\mathbf{Set})$ as,

$$X \mapsto X \times M,$$

the usual direct product of sets, and for each $f \in \hom_{\mathbf{Set}}(X, Y)$,

$$f \mapsto f \times M$$

where,

$$\begin{aligned} f \times M \colon X \times M &\to Y \times M, \\ (x, m) &\mapsto (f(x), m). \end{aligned}$$

Recall we also have the functor h_M defined in Lemma 3.12.

We can define the natural transformation $\eta \colon id_{\mathbf{Set}} \Rightarrow h_M((-) \times M)$ where for each $X \in \mathrm{Ob}(\mathbf{Set})$,

$$\begin{aligned} \eta_X \colon X &\to h_M(X \times M), \\ x &\mapsto \eta_X^{(x)} \end{aligned}$$

where,

$$\begin{aligned} \eta_X^{(x)} \colon M &\to X \times M, \\ m &\mapsto (x, m). \end{aligned}$$

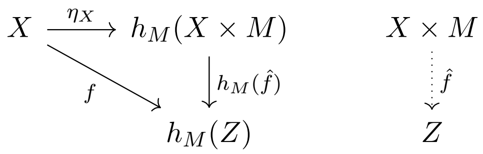
Then we have a unique function defined

$$\begin{aligned} \hat{f} \colon X \times M &\to Z, \\ (x, m) &\mapsto f_x(m) \end{aligned}$$

where,

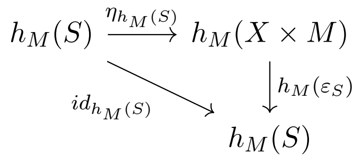
$$\begin{aligned} f_x \colon M &\to Z, \\ m &\mapsto (f(x))(m) \end{aligned}$$

for which the diagram on the left,



commutes.

The co-unit $\varepsilon\colon (h_M(-)) \times M \Rightarrow id_{\mathbf{Set}}$ is defined for each $S \in \mathrm{Ob}(\mathbf{Set})$ as the unique morphism which,



commutes. Therefore define,

$$\begin{aligned}\varepsilon_S\colon (h_M(S)) \times M &\to S,\\ (g_t, m) &\mapsto g_t(m)\end{aligned}$$

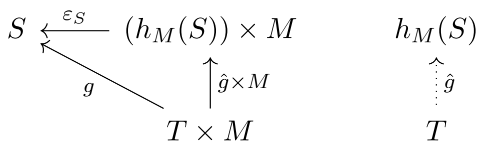
then for each $g \in \hom_{\mathbf{Set}}(S, T \times M)$ there is a unique function given,

$$\begin{aligned}\hat{g}\colon T &\to h_M(S)\\ t &\mapsto g_t\end{aligned}$$

where,

$$\begin{aligned}g_t\colon M &\to S,\\ g_t(m) &\mapsto g(t, m)\end{aligned}$$

for which the diagram on the left,



commutes.

Now we can classify all other adjoint situations by looking at bijections; Let M' be a set such that there exists a bijection $\tau\colon M \to M'$. τ can be realised as a natural transformation

$$\tau\colon (-) \times M \Rightarrow (-) \times M'$$

where for each $X \in \mathrm{Ob}(\mathbf{Set})$ we have,

$$\begin{aligned}\tau_X\colon X \times M &\to X \times M',\\ (x, m) &\mapsto (x, \tau(m)).\end{aligned}$$

We show the naturality condition holds. Let $X, Y \in \mathrm{Ob}(\mathbf{Set})$ and $f \in \hom_{\mathbf{Set}}(X, Y)$ then for $(x, m) \in X \times M$,

$$\begin{aligned}(\tau_Y \circ f \times M)((x, m)) &= (\tau_Y((f(x), m)))\\ &= (f(x), \tau(m))\\ &= f \times M((x, \tau(m)))\\ &= f \times M \circ \tau_X((x, m))\end{aligned}$$

Therefore the naturality condition holds. τ has a natural inverse

$$\tau^{-1} \colon (-) \times M' \Rightarrow (-) \times M$$

where for each $X \in \mathrm{Ob}(\mathbf{Set})$ we have,

$$\begin{aligned} \tau_X^{-1} \colon X \times M &\to X \times M', \\ (x, m') &\mapsto (x, \tau^{-1}(m')). \end{aligned}$$

since τ is a bijection. Hence τ is a natural isomorphism.

So for the functor h_M we have adjoint situations $(\tau((-) \times M), h_M, h_M\tau \circ \eta, \varepsilon \circ \tau^{-1} h_M)$.

6 Initial and terminal objects

In this section we use notation and definitions from Leinster [5].

Definition 6.1. Let $\mathscr{C}$ be a category. We call an $\mathscr{C} - Objects$, $I \in \mathrm{Ob}(\mathscr{C})$ *initial* if for any object $C \in \mathrm{Ob}(\mathscr{C})$, there is exactly one morphism $f \in \hom_{\mathscr{C}}(I, C)$. We call an $\mathscr{C} - Objects$, $T \in \mathrm{Ob}(\mathscr{C})$ *terminal* if for any object $C \in \mathrm{Ob}(\mathscr{C})$, there is exactly one morphism $f \in \hom_{\mathscr{C}}(C, T)$.

The following Lemma 6.2 and proof is adapted from Leinster [5] (page 49 Lemma 2.1.8)

Lemma 6.2. Let $\mathscr{C}$ be a category and $I, I' \in \mathrm{Ob}(\mathscr{C})$ initial objects in $\mathscr{C}$. There exists a unique isomorphism $f \in \hom_{\mathscr{C}}(I, I')$. Let $T, T' \in \mathrm{Ob}(\mathscr{C})$ be terminal objects in $\mathscr{C}$ then there exists a unique isomorphism $g \in \hom_{\mathscr{C}}(T, T')$.

Proof. Since I is initial there exists a unique morphism $f \in \hom_{\mathscr{C}}(I, I')$.

It remains to show f is an isomorphism. Since I' is initial then there exists a unique morphism $f' \in \hom_{\mathscr{C}}(I', I)$. Therefore $f' \circ f \in \hom_{\mathscr{C}}(I, I)$ but since I is initial $f' \circ f$ is the unique morphism in $\hom_{\mathscr{C}}(I, I)$. Since $\mathscr{C}$ is a category we have $id_I \in \hom_{\mathscr{C}}(I, I)$, hence id_I is the unique morphism in $\hom_{\mathscr{C}}(I, I)$ therefore,

$$f \circ f' = id_I$$

similarly,

$$f' \circ f = id_{I'}$$

since $f \circ f' \in \hom_{\mathscr{C}}(I', I')$ is unique. Therefore f is an isomorphism.

Since T is terminal there exists a unique morphism $g \in \hom_{\mathscr{C}}(T', T)$, also since T' is terminal there exists a unique morphism $g' \hom_{\mathscr{C}}(T, T')$. Then we have,

$$g \circ g' = id_{T'}$$

since $g \circ g' \in \hom_{\mathscr{C}}(T', T')$ is unique. Similarly,

$$g' \circ g = id_T$$

since $g' \circ g \hom_{\mathscr{C}}(T, T)$ is unique. □

Remark 6.3. Given a category there may or may not exist an initial or terminal objects but the above Lemma 6.2 states that if there are initial objects they are all isomorphic and if there are terminal objects they are all isomorphic.

6.1 Examples

The following example can be found on nCatLab[8].

Example 6.4. Let **Set** be the category of sets. The empty set $\varnothing \in \mathrm{Ob}(\mathbf{Set})$ is initial. Any singleton set $\{x\} \in \mathrm{Ob}(\mathbf{Set})$ is terminal. For any set $Y \in \mathrm{Ob}(\mathbf{Set})$ we have the unique map

$$f \colon Y \to \{x\},$$
$$y \mapsto x.$$

hence $\{x\}$ is terminal. There also exists a unique function $\emptyset \colon \varnothing \to Y$ which is the empty function, hence $\varnothing$ is initial.

Example 6.5. Let **Grp** be the category of groups. The trivial group $\{id\}$ is initial and terminal. Given any group $(G, \circ)$ There exists a group homomorphism,

$$\begin{aligned} f \colon \{id\} &\to (G, \circ), \\ id &\mapsto id_G, \end{aligned}$$

and also a group homomorphism,

$$\begin{aligned} f \colon (G, \circ) &\to \{id\}, \\ x &\mapsto id. \end{aligned}$$

Both of these functions are unique since group homomorphisms must preserve identities.

6.2 Initial and terminal objects as adjoint functors

Definition 6.6. We call the category **1** with one object $1 \in \mathrm{Ob}(\mathbf{1})$ and one morphism $id_1 \in \hom_{\mathbf{1}}(1, 1)$ the *identity category*.

The following Lemma 6.9 is adapted from Leinster [5] (Page 49 Example 2.1.9)

Lemma 6.7. Let $\mathscr{C}$ be a category. Given $C \in \mathrm{Ob}(\mathscr{C})$, there exists a functor $\mathrm{I_C} \colon \mathscr{C} \to \mathbf{1}$ which maps

$$1 \mapsto C$$

and,

$$id_1 \mapsto id_C.$$

Further, there exists a functor $\mathrm{T} \colon \mathbf{1} \to \mathscr{C}$ where for $C \in \mathrm{Ob}(\mathscr{C})$,

$$C \mapsto 1$$

and for $f \in \hom_{\mathscr{C}}(A, B)$,

$$f \mapsto id_1.$$

Proof. I_C is a functor since,

$$\begin{aligned} I_C(id_1) &= id_C \\ &= id_{I_C(1)} \end{aligned}$$

and,

$$\begin{aligned} I_C(id_1 \circ id_1) &= I_C(id_1) \\ &= id_C \\ &= id_C \circ id_C \\ &= I_C(id_1) \circ I_C(id_1). \end{aligned}$$

T is a functor since for all $C \in \mathrm{Ob}(\mathscr{C})$,

$$T(id_C) = id_1 = id_{T(1)}$$

and for any $f \in \hom_{\mathscr{C}}(A, B)$ and $g \in \hom_{\mathscr{C}}(B, C)$,

$$\begin{aligned} T(g \circ f) &= id_1 \\ &= id_1 \circ id_1 \\ &= T(g) \circ T(f). \end{aligned}$$

$\square$

Lemma 6.8. Given a category $\mathscr{C}$ let I be the set of all functors of the form I_C as in Lemma 6.7, there exists a bijection

$$\begin{aligned}\psi \colon \mathrm{Ob}(\mathscr{C}) &\to I,\\ C &\mapsto I_C.\end{aligned}$$

Proof. Given $I_C \in I$ there exists $C \in \mathrm{Ob}(\mathscr{C})$ such that $C \mapsto I_C$ by Lemma 6.7, hence ψ is surjective.

Given $I_C = I_{C'}$ then $C = C'$ by Lemma 6.7 hence ψ is injective.

Therefore ψ is a bijection. □

Lemma 6.9. Let $\mathscr{C}$ be a category and $C \in \mathscr{C}$ then:

1. $\mathrm{I_C} \colon \mathbf{1} \to \mathscr{C}$ is left adjoint if and only if C is an initial object of $\mathscr{C}$;
2. $\mathrm{I_C} \colon \mathbf{1} \to \mathscr{C}$ is right adjoint if and only if C is a terminal object of $\mathscr{C}$.

Proof. (1): Let I_C be a left adjoint. Then there exists an adjoint situation defined in Theorem 5.6. Hence there is a unique functor $\mathrm{T}^* \colon \mathscr{C} \to \mathbf{1}$ but since the only functor from $\mathscr{C}$ to $\mathbf{1}$ is the one defined in Lemma 6.7 we have $T^* = T$. Therefore, for each $C' \in \mathrm{Ob}(\mathscr{C})$,

$$\begin{aligned}&\hom_{\mathscr{C}}(I_C(1), C') \cong \hom_{\mathbf{1}}(1, T(C'))\\ \implies &\hom_{\mathscr{C}}(C, C') \cong \hom_{\mathbf{1}}(1, 1)\\ \implies &\hom_{\mathscr{C}}(C, C') \cong \{id_1\}.\end{aligned}$$

Therefore C is an initial object.

Let $C \in \mathrm{Ob}(\mathscr{C})$ be an initial object, therefore we have for any $C' \in \mathrm{Ob}(\mathscr{C})$

$$\hom_{\mathscr{C}}(C, C').$$

has one element. Then we have,

$$\begin{aligned}&\hom(C, C') \cong \{id_1\}\\ \implies &\hom(C, C') \cong \hom_{\mathbf{1}}(1, 1)\\ \implies &\hom_{\mathscr{C}}(I_C(1), C') \cong \hom_{\mathbf{1}}(1, T(C')).\end{aligned}$$

Therefore by 5.6 I_C is a left adjoint.

(2): Let I_C be a right adjoint. Then there exists an adjoint situation defined in Theorem 5.6. Hence there is a unique functor $\mathrm{T}^* \colon \mathscr{C} \to \mathbf{1}$ but since the only functor from $\mathscr{C}$ to $\mathbf{1}$ is the one defined in Lemma 6.7 we have $T^* = T$. Therefore, for each $C' \in \mathrm{Ob}(\mathscr{C})$,

$$\begin{aligned}&\hom_{\mathbf{1}}(T(C'), 1) \cong \hom_{\mathscr{C}}(C', I_C(1)),\\ \implies &\hom_{\mathbf{1}}(1, 1) \cong \hom_{\mathscr{C}}(C', C),\\ \implies &\{id_1\} \cong \hom_{\mathscr{C}}(C', C).\end{aligned}$$

Therefore C is a terminal object. Let $C \in \mathrm{Ob}(\mathscr{C})$ be a terminal object, therefore we have for any $C' \in \mathrm{Ob}(\mathscr{C})$

$$\hom_{\mathscr{C}}(C', C).$$

has one element. Then we have,

$$\begin{aligned}&\{id_1\} \cong \hom_{\mathscr{C}}(C', C)\\ \implies &\hom_{\mathbf{1}}(1, 1) \cong \hom_{\mathscr{C}}(C', C)\\ \implies &\hom_{\mathbf{1}}(T(C'), 1) \cong \hom_{\mathscr{C}}(C', I_C(1)).\end{aligned}$$

Therefore by 5.6 I_C is a right adjoint. □

Example 6.10. Let $\mathbf{Set}$ be the category of sets. We have the functor,

$$\begin{aligned} \mathrm{I}_{\varnothing} \colon \mathbf{1} &\to \mathbf{Set}, \\ 1 &\mapsto \varnothing. \end{aligned}$$

Then for $X \in \mathrm{Ob}(\mathbf{Set})$, we have a universal morphism (η_X, C_X), where $C_X = 1$ and $\eta_X = \emptyset$, the empty function:

$$\eta_X \colon X \mapsto \varnothing$$

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & I_{\varnothing}(1) = \varnothing & & 1 \\ & \searrow^{\emptyset} & \big\downarrow {\scriptstyle I_{\varnothing}(id_1) = id_{\varnothing}} & & \vdots\, {\scriptstyle id_1} \\ & & I_{\varnothing}(1) = \varnothing & & 1 \end{array} .$$

Therefore, $I_{\varnothing}$ is right adjoint.

7 Monads

Monads on a category $\mathscr{C}$ are endofunctors of $\mathscr{C}$ with some extra structure. We can look at monads to study adjunctions since for adjoint functors F and G we will show FG is a monad. Conversely, we will find that given a monad there are suitable adjoint functors which define the monad, these however are not unique.

Definition 7.1. Let $\mathscr{C}$ be a category. A *monad on* $\mathscr{C}$ is a triple (T, η, μ) where $\mathrm{T}\colon \mathscr{C} \to \mathscr{C}$ is endofunctor, $\eta\colon id_{\mathscr{C}} \Rightarrow T$ and $\mu\colon T^2 \Rightarrow T$ are natural transformations for which the following conditions hold:

$$\mu \circ T\mu = \mu \circ \mu T;$$
$$\mu \circ T\eta = \mu \circ \eta T = id_T$$

where id_T is the identity natural transformation on F as defined in Definition 4.15. $\circ$ is vertical composition of natural transformations as in Definition 4.6. These conditions are equivalent to saying the diagrams,

$$\begin{array}{ccc} T^3 & \overset{T\mu}{\Longrightarrow} & T^2 \\ {\scriptstyle \mu T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T^2 & \underset{\mu}{\Longrightarrow} & T \end{array} \qquad \begin{array}{ccc} T & \overset{\eta T}{\Longrightarrow} & T^2 \\ {\scriptstyle T\eta}\Downarrow & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle \mu} \\ T^2 & \underset{\mu}{\Longrightarrow} & T \end{array}$$

commute. That is, for each object, $X \in \mathrm{Ob}(\mathscr{C})$, the diagram,

$$\begin{array}{ccc} T^3(X) & \xrightarrow{T(\mu_X)} & T^2(X) \\ {\scriptstyle \mu_{T(X)}}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ T^2(X) & \xrightarrow[\mu_X]{} & T(X) \end{array} \tag{7.1}$$

and,

$$\begin{array}{ccc} T(X) & \xrightarrow{\eta_{T(X)}} & T^2(X) \\ {\scriptstyle T(\eta_X)}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu_X} \\ T^2(X) & \xrightarrow[\mu_X]{} & T(X) \end{array} \tag{7.2}$$

commutes.

The following example is stated in [1] (Page 318, Examples 20.2 (3)), here we add the proof.

Example 7.2. Let $\mathrm{P}\colon \mathbf{Set} \to \mathbf{Set}$ be the power set functor defined in Example 3.17. Let $\eta\colon id_{\mathbf{Set}} \Rightarrow P$ be defined as in Example 4.2 and $\mu\colon P^2 \Rightarrow P$ be the natural transformation as defined in Example 4.3. Then $\mathbf{P} = (P, \eta, \mu)$ is a monad.

Proof. We need to show that the diagrams

$$\begin{array}{ccc} P^3 & \xrightarrow{P\mu} & P^2 \\ {\scriptstyle \mu P}\downarrow & & \downarrow{\scriptstyle \mu} \\ P^2 & \xrightarrow[\mu]{} & P \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\eta P} & P^2 \\ {\scriptstyle P\eta}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} \\ P^2 & \xrightarrow[\mu]{} & P \end{array}$$

commute. For each $X \in \mathrm{Ob}(\mathbf{Set})$ and each element $A \in P^3(X)$ we have,

$$\begin{aligned}
(\mu \circ P\mu)_X(A) &= \mu_X \circ P(\mu_X)(A) \\
&= \mu_X(\mu_X[A]) \\
&= \mu_X(\{\mu_X(a) | a \in A\}) \\
&= \bigcup_{a \in A} \mu_X(a) \\
&= \bigcup_{a \in A} \bigcup_{a' \in a} a' \\
&= \bigcup_{a' \in \bigcup_{a \in A} a} a' \\
&= \mu_X(\bigcup_{a \in A} a) \\
&= \mu_X(\mu_{P(X)}(A)) \\
&= (\mu \circ \mu P)_X(A).
\end{aligned}$$

Also for each $B \in P(X)$,

$$\begin{aligned}
(\mu \circ P\eta)_X(B) &= \mu_X \circ P(\eta_X(B)) \\
&= \mu_X(\eta_X[B]) \\
&= \mu_X(\{\eta_X(b) | b \in B\}) \\
&= \bigcup_{b \in B} \eta_X(b) \\
&= \bigcup_{b \in B} \{B\} \\
&= B \\
&= id_{P(X)}(B) \\
&= \bigcup_{b' \in \{B\}} \\
&= \mu_X(\{B\}) \\
&= \mu_X(\eta_{P(X)}(B)) \\
&= (\mu \circ \eta P)_X(B).
\end{aligned}$$

Therefore $\mathbf{P}$ is a monad. □

The following example was briefly stated in the Wikipedia articles on monads [17], here we will go into more detail.

Example 7.3. Let $\mathscr{C}$ be a category, the identity functor $\mathrm{id}_{\mathscr{C}} \colon \mathscr{C} \to \mathscr{C}$ as defined in Definition 3.15 forms a monad with $\eta \colon id_{\mathscr{C}} \Rightarrow id_{\mathscr{C}}$ defined for each $X \in \mathrm{Ob}(\mathscr{C})$,

$$\eta_X = id_X \in \hom_{\mathscr{C}}(X, X)$$

and, $\mu \colon id_{\mathscr{C}} \Rightarrow id_{\mathscr{C}}$ defined for each $X \in \mathrm{Ob}(\mathscr{C})$,

$$\mu_X = id_X \in \hom_{\mathscr{C}}(X, X).$$

Proof. We have for each $X \in \mathrm{Ob}(\mathscr{C})$,

$$\begin{aligned}(\mu \cdot F\mu)_X &= \mu_X \circ F(\mu_X)\\ &= id_X\\ &= \mu_X \circ \mu_{F(X)}\\ &= (\mu \cdot \mu F)_X\end{aligned}$$

and,

$$\begin{aligned}(\mu \cdot F\eta)_X &= \mu_X \circ F(\eta_X)\\ &= id_X\\ &= \mu_X \circ \eta_{F(X)}\\ &= (\mu \cdot \eta F)_X.\end{aligned}$$

Therefore we have a monad. □

The following example 7.4 follows the example given in The Catsters video series on monads [2].

Example 7.4. Let **Set** be the category of sets, defined in Example 2.4. Let $\mathrm{F}\colon \mathbf{Set} \to \mathbf{Mon}$ be the free monoid functor defined in Definition 3.9 and $\mathrm{U}\colon \mathbf{Mon} \to \mathbf{Set}$ be the forgetful monoid functor as defined in Definition 3.4. We define the triple $\mathbf{M} = (T, \eta, \mu)$ where $\mathrm{T} = \mathrm{UF}\colon \mathbf{Set} \to \mathbf{Set}$ for each set $X \in \mathrm{Ob}(\mathbf{Set})$,

$$\begin{aligned}\eta_X \colon X &\to \gamma(X),\\ x &\mapsto (x).\end{aligned}$$

Where $\gamma(X)$ is the set of words of X, and

$$\mu_X \colon \gamma(\gamma(X)) \to \gamma(X)$$

be such that,

$$\begin{aligned}&\big((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_m), \dots\dots, (z_1, z_2, \dots, z_r)\big)\\ &\mapsto \big(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m, \dots\dots, z_1, z_2, \dots, z_r\big).\end{aligned}$$

(Note that $\gamma(\gamma(X))$ is the set of words of words of X). We claim that **M** is a monad.

T is a functor since it is a composition of two functors. We first show η and μ are natural transformations.

First we prove the naturality of η: Let $f \in \hom_{\mathscr{S}}(X, X')$ then we have,

$$\begin{aligned}(\eta_{X'} \circ f)(x) &= (f(x))\\ &= T(f)((x))\\ &= (T(f) \circ \eta_X)(x).\end{aligned}$$

Therefore the naturality condition holds and η is a natural transformation.

Now let us prove the naturality of μ: Let $f \in \hom_{\mathbf{Set}}(X, X')$ then we have,

$$\begin{aligned}&(\mu_{X'} \circ T(T(f)))\big(((x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_m), \dots\dots, (z_1, z_2, \dots, z_r))\big)\\ &= \mu_{X'}\big(((f(x_1), f(x_2), \dots, f(x_n))(f(y_1), f(y_2), \dots, f(y_m)), \dots\dots, (f(z_1), f(z_2), \dots, f(z_r)))\big)\\ &= \big(f(x_1), f(x_2), \dots, f(x_n), f(y_1), f(y_2), \dots, f(y_m), \dots\dots, f(z_1), f(z_2), \dots, f(z_r)\big)\\ &= T(f)\big((x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m, \dots\dots, z_1, z_2, \dots, z_r)\big)\\ &= (T(f) \circ \mu_X)\big(((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_m), \dots\dots, (z_1, z_2, \dots, z_r))\big).\end{aligned}$$

Therefore the naturality condition holds and μ is a natural transformation.

We now prove that the diagram (7.1) commutes. Let

$$\begin{pmatrix} ((a,\dots,a_n),(b,\dots,b_m),\dots,(c,\dots,c_r)), \\ ((a',\dots,a'_{n'}),(b',\dots,b'_{m'}),\dots,(c',\dots,c'_{r'})), \\ \dots, \\ ((a'',\dots,a''_{n''}),(b'',\dots,b''_{m''}),\dots,(c'',\dots,c''_{r''})) \end{pmatrix} \in \gamma(\gamma(\gamma(X)))$$

be a general word of a word of a word of X. Here we have extended the notation in to multi line, where each row is a word of a word of X.

We have,

$$(\mu \circ T\mu)_X(\begin{pmatrix} ((a,\dots,a_n),(b,\dots,b_m),\dots,(c,\dots,c_r)), \\ ((a',\dots,a'_{n'}),(b',\dots,b'_{m'}),\dots,(c',\dots,c'_{r'})), \\ \dots, \\ ((a'',\dots,a''_{n''}),(b'',\dots,b''_{m''}),\dots,(c'',\dots,c''_{r''})) \end{pmatrix})$$

$$= (\mu_X \circ T(\mu_X))(\begin{pmatrix} ((a,\dots,a_n),(b,\dots,b_m),\dots,(c,\dots,c_r)), \\ ((a',\dots,a'_{n'}),(b',\dots,b'_{m'}),\dots,(c',\dots,c'_{r'})), \\ \dots, \\ ((a'',\dots,a''_{n''}),(b'',\dots,b''_{m''}),\dots,(c'',\dots,c''_{r''})) \end{pmatrix})$$

$$= \mu_X(\begin{pmatrix} (a,\dots,a_n,b,\dots,b_m,\dots,c,\dots,c_r), \\ (a',\dots,a'_{n'},b',\dots,b'_{m'},\dots,c',\dots,c'_{r'}), \\ \dots, \\ (a'',\dots,a''_{n''},b'',\dots,b''_{m''},\dots,c'',\dots,c''_{r''}) \end{pmatrix})$$

$$= \begin{pmatrix} a,\dots,a_n,b,\dots,b_m,\dots,c,\dots,c_r, \\ a',\dots,a'_{n'},b',\dots,b'_{m'},\dots,c',\dots,c'_{r'}, \\ \dots, \\ a'',\dots,a''_{n''},b'',\dots,b''_{m''},\dots,c'',\dots,c''_{r''} \end{pmatrix}$$

$$= \mu_X(\begin{pmatrix} (a,\dots,a_n),(b,\dots,b_m),\dots,(c,\dots,c_r), \\ (a',\dots,a'_{n'}),(b',\dots,b'_{m'}),\dots,(c',\dots,c'_{r'}), \\ \dots, \\ (a'',\dots,a''_{n''}),(b'',\dots,b''_{m''}),\dots,(c'',\dots,c''_{r''}) \end{pmatrix})$$

$$= (\mu_X \circ \mu_{T(X)})(\begin{pmatrix} ((a,\dots,a_n),(b,\dots,b_m),\dots,(c,\dots,c_r)), \\ ((a',\dots,a'_{n'}),(b',\dots,b'_{m'}),\dots,(c',\dots,c'_{r'})), \\ \dots, \\ ((a'',\dots,a''_{n''}),(b'',\dots,b''_{m''}),\dots,(c'',\dots,c''_{r''})) \end{pmatrix})$$

$$= (\mu \circ \mu T)_X(\begin{pmatrix} ((a,\dots,a_n),(b,\dots,b_m),\dots,(c,\dots,c_r)), \\ ((a',\dots,a'_{n'}),(b',\dots,b'_{m'}),\dots,(c',\dots,c'_{r'})), \\ \dots, \\ ((a'',\dots,a''_{n''}),(b'',\dots,b''_{m''}),\dots,(c'',\dots,c''_{r''})) \end{pmatrix})$$

To show the diagram (7.2) commutes. Let $(x_1, x_2, \dots, x_n) \in \gamma(X)$ then we have,

$$\begin{aligned} (\mu \circ T\eta)_X((x_1,x_2,\dots,x_n)) &= \mu_X(T(\eta_X)((x_1,x_2,\dots,x_n))) \\ &= \mu_X(((x_1),(x_2),\dots,(x_n))) \\ &= (x_1,x_2,\dots,x_n) \\ &= \mu_X(((x_1,x_2,\dots,x_n))) \\ &= \mu_X(\eta_{T(X)}((x_1,x_2,\dots,x_n))) \\ &= (\mu \circ \eta T)_X((x_1,x_2,\dots,x_n)). \end{aligned}$$

Therefore, $\mathbf{M}$ is a monad.

7.1 Algebras of monads

The following definition is adapted from Adámek - Herrlick - Strecker [1] (Page 318, Definition 20.4) and the Casters video series on Monads [2]

Definition 7.5. Let $\mathscr{C}$ be a category and $\mathbf{T} = (T, \eta, \mu)$ be a monad on $\mathscr{C}$. An *algebra of* $\mathbf{T}$ is a pair, (X, θ), where $X \in \mathrm{Ob}(\mathscr{C})$ and $\theta \in \hom_{\mathscr{C}}(T(X), X)$, such that the following axioms hold:

1. $\theta \circ \eta_X = id_X$;
2. $\theta \circ T\theta = \theta \circ \mu_X$.

That is the diagrams,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T(X) \\ & {\scriptstyle id_X}\searrow & \downarrow{\scriptstyle \theta} \\ & & X \end{array} \tag{7.3}$$

and,

$$\begin{array}{ccc} T^2(X) & \xrightarrow{T\theta} & T(X) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \theta} \\ T(X) & \xrightarrow[\theta]{} & X \end{array} \tag{7.4}$$

commute.

Definition 7.6. Let $\mathscr{C}$ be a category, $\mathbf{T} = (T, \eta, \mu)$ be a monad on $\mathscr{C}$ and, (A, θ) and (B, ϕ) algebras of $\mathbf{T}$. A *morphism of algebras of* T is a morphism $f \in \hom_{\mathscr{C}}(A, B)$ such that

$$\phi \circ T(f) = f \circ \theta$$

that is the following diagram,

$$\begin{array}{ccc} T(A) & \xrightarrow{T(f)} & T(B) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi} \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

Remark 7.7. The collection of morphisms of algebras, between the same pair of algebras, is a set if we assume the definition of a category in Definition 2.1 since each $\hom$ set of $\mathscr{C}$ is a set.

Lemma 7.8. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad of a category $\mathscr{C}$. Then for each $X \in \mathrm{Ob}(\mathscr{C})$, $(T(X), \mu_X)$ is an algebra.

Proof. First note that $\eta_X \in \hom_{\mathscr{C}}(T(T(X)), T(X))$ and therefore is a morphism $\eta_X \in \hom_{\mathscr{C}}(T(Y), Y)$ where $Y = T(X) \in \mathrm{Ob}(\mathscr{C})$ so we check the axioms in Definition 7.5.

$$\begin{aligned} \mu_X \circ \eta_Y &= \mu_X \circ \eta_{T(X)} \\ &= \mu_X \circ (\eta T)_X \\ &= id_{T(X)}, && \text{By Definition 7.1} \\ &= id_Y \end{aligned}$$

and,

$$\begin{aligned}\mu_X \circ T(\mu_X) &= \mu_X \circ T(\mu_X) \\ &= \mu_X \circ \mu_{T(X)}, && \text{By Definition 7.1} \\ &= \mu_X \circ \mu_Y.\end{aligned}$$

Therefore, (X, μ_X) is an algebra on $\mathbf{T}$. □

Definition 7.9. For a monad $\mathbf{T} = (T, \eta, \mu)$ we call $(T(X), \mu_X)$ the *free* $\mathbf{T}$*-algebra on* X.

The following Lemma 7.10 is stated in Adámek - Herrlick - Strecker [1] (Page 318, Definition 20.4) but not proven.

Lemma 7.10. Let $\mathscr{C}$ be a category and $\mathbf{T} = (T, \eta, \mu)$ a monad over $\mathscr{C}$. The collection of all algebras, written $\mathrm{Alg}(\mathbf{T})$, forms a category with morphisms of algebras, called the *Eilenberg-Moore category*. Composition is given by the composition of the category and for each $(A, \theta) \in \mathrm{Ob}(\mathrm{Alg}(\mathbf{T}))$ the identity is the id_A.

Proof. We first show composition of two morphisms for algebras is a morphism for algebras. Let $(A, \theta), (B, \phi), (C, \psi)) \in \mathrm{Ob}(\mathrm{Alg}(\mathbf{T}))$, $f \in \hom_{\mathscr{C}}(A, B)$ and $g \in \hom_{\mathscr{C}}(B, C)$ be morphism for algebras then $g \circ f \in \hom_{\mathscr{C}}(A, C)$ and we have,

$$\begin{aligned}\psi \circ T(g \circ f) &= \psi \circ T(g) \circ T(f) \\ &= g \circ \phi \circ T(f) \\ &= g \circ f \circ \theta.\end{aligned}$$

Therefore, $g \circ f$ is a morphism of algebras.

Composition of these algebras is associative since composition in $\mathscr{C}$ is associative.

For an algebra (A, θ) and $id_A \in \hom_{\mathscr{C}}(A, A)$ we have,

$$\theta \circ id_A = \theta = id_A \circ \theta$$

Hence, id_A is a morphism for algebras. For a morphism of algebras $f \in \hom_{\mathrm{Alg}(\mathbf{T})}((A, \theta_A), (B, \theta_B))$ we have,

$$f \circ id_A = f = id_B \circ f.$$

Hence for each $(A, \theta_A) \in \mathrm{Ob}(\mathrm{Alg}(\mathbf{T}))$, id_A is an identity morphism. Therefore we have a category. □

7.1.1 Examples

The following example is stated in Adámek - Herrlick - Strecker [1] (Page 318, Example 20.5 (1)).

Example 7.11. Let $\mathbf{id}_{\mathscr{C}} = (id_{\mathscr{C}}, \eta, \mu)$ be the identity monad as defined in Example 7.3. Then an algebra on $\mathbf{id}_{\mathscr{C}}$ is an object $X \in \mathrm{Ob}(\mathscr{C})$ and a morphism $f \in \hom_{\mathscr{C}}(X, X)$ such that the diagrams (7.3) and (7.4) commute.

We have,

$$f \circ \eta_X = f \circ id_X = id_X$$

therefore, $f = id_X$ and,

$$\begin{aligned}f \circ T(f) &= id_X \circ id_X \\ &= f \circ \mu_X.\end{aligned}$$

Hence, algebras are of the form (X, id_X) for all $X \in \mathrm{Ob}(\mathscr{C})$.

A morphism of algebras for two algebras (X, id_X) and (Y, id_Y) is a morphism, $f \in \hom_{\mathscr{C}}(X, Y)$ such that

$$id_Y \circ f = f \circ id_X.$$

Therefore f is any morphism in $\mathscr{C}$. So the category $\mathrm{Alg}(\mathbf{id}_{\mathscr{C}})) \cong \mathscr{C}$

The following Example 7.12 is stated in Adámek - Herrlick - Strecker [1] (Page 318, Example 20.5 (2)).

Example 7.12. Let $\mathbf{M} = (T, \eta, \mu)$ be the monad defined in Example 7.4. An algebra of $\mathbf{M}$, (X, θ) is a set $X \in \mathrm{Ob}(\mathbf{Set})$ and a morphism $\theta \in \hom_{\mathbf{Set}}(\gamma(X), X)$ such that the diagrams (7.4) and (7.3) commute. Hence we have for $x \in X$,

$$\begin{aligned}(\theta \circ \eta_X)(x) &= \theta((x)) \\ &= id_X(x)\end{aligned}$$

therefore, $x = \theta((x))$.

For a general word of a word of X,

$$\gamma(\gamma(x)) = ((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_m), \ldots\ldots, (z_1, z_2, \ldots, z_r))$$

we have,

$$\begin{aligned}(\theta \circ T\theta)(\gamma(\gamma(x))) &= \theta(\theta((x_1, \ldots, x_n)), \theta((y_1, \ldots, y_m)), \ldots, \theta((z_1, \ldots, z_r)))) \\ &= \theta((x_1, \ldots, z_r)) \\ &= (\theta \circ \mu_X)(\gamma(\gamma(x))).\end{aligned}$$

Therefore, an algebra of $\mathbf{M}$ (X, θ) defines a monoid $(X, (- \cdot -), e)$ where $e = ()$ is the empty word and composition is defined,

$$\begin{aligned}(- \cdot -) \colon X \times X &\to X, \\ x \cdot y &\mapsto \theta((x, y)).\end{aligned}$$

$e = ()$ is an identity since for any $x \in X$,

$$x \cdot e = \theta((x)) = x$$

and $(- \cdot -)$ is associative since given $x, y, z \in X$,

$$\begin{aligned}(x \cdot y) \cdot z &= \theta((x, y)) \cdot z \\ &= \theta((\theta((x, y)), z)) \\ &= \theta(x, y, z)\end{aligned}$$

by the commutativity of diagram (7.4) and,

$$\begin{aligned}x \cdot (y \cdot z) &= x \cdot \theta((x, y)) \\ &= \theta((x, \theta(x, y))) \\ &= \theta((x, y, z))\end{aligned}$$

by the commutativity of diagram (7.4). Hence

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

Therefore each algebra of $\mathbf{M}$ forms a monoid hence $\mathrm{Alg}(\mathbf{M})$ is a subcategory of $\mathbf{Mon}$.

To show that $\mathrm{Alg}(\mathbf{M}) \cong \mathbf{Mon}$ we show every monoid can be defined by an algebra of $\mathbf{M}$. Let $(X, \cdot, e)$ be a monoid then we can define the function θ as,

$$\theta \colon \gamma(X) \to X$$
$$(x_1, x_2, \dots, x_n) \mapsto x_1 \cdot x_2 \cdot \dots \cdot x_n.$$

Then we check (X, θ) is an algebra of $\mathbf{M}$. For an element $x \in X$

$$\begin{aligned}(\theta \circ \eta_X)(x) &= \theta((x)) \\ &= x\end{aligned}$$

Therefore, diagram (7.3) commutes. For a general word of a word of X, $\gamma(\gamma(x))$,

$$\begin{aligned}(\theta \circ T\theta)(\gamma(\gamma(x))) &= \theta(((x_1 \cdot x_2 \cdot \dots \cdot x_n), (y_1 \cdot \dots \cdot y_m), \dots, (z_1 \cdot \dots \cdot z_r))) \\ &= x_1 \cdot \dots \cdot z_r \\ &= \theta((x_1, \dots, z_r)) \\ &= \theta(\mu_x(\gamma(\gamma(x)))) \\ &= (\theta \circ \mu_X)(\gamma(\gamma(x))).\end{aligned}$$

Therefore, the diagram (7.4) commutes and hence (X, θ) is an algebra of $\mathbf{M}$.

This means (X, θ) an algebra of $\mathbf{M}$ if and only if $(X, \cdot, e)$ is a monoid. Hence there is a one to one correspondence between monoids and algebras of $\mathbf{M}$.

7.2 Adjoint Functors as monads

The following lemma can be found in Adámek - Herrlick - Strecker [1] (Page 318, Proposition 20.3). Where here we add the completed proof which was left as an exercise.

Lemma 7.13. Let $\mathrm{F} \colon \mathscr{D} \to \mathscr{C}$ and $\mathrm{G} \colon \mathscr{C} \to \mathscr{D}$ be functors and $(F, G, \eta, \varepsilon)$ be an adjoint situation. We have an endofunctor, $\mathrm{T} = \mathrm{GF} \colon \mathscr{D} \to \mathscr{D}$ and natural transformation $G\varepsilon F \colon T^2 \Rightarrow T$. Then $(GF, \eta, G\varepsilon F)$ is a monad on $\mathscr{D}$.

Proof. Since $\varepsilon \colon FU \Rightarrow id_{\mathbf{Mon}}$ is a natural transformation for each $M \in \mathrm{Ob}(\mathbf{Mon})$ we have,

$$\begin{aligned}(\varepsilon \circ FG\varepsilon)_M &= \varepsilon_M \circ F(G(\varepsilon_M)), \\ &= \varepsilon_{FG(M)} \circ \varepsilon_M, \\ &= (\varepsilon \circ \varepsilon FG).\end{aligned}$$

Therefore, for each $X \in \mathscr{D}$,

$$\begin{aligned}(\mu \circ T\mu)_X &= (G\varepsilon F \circ GFG\varepsilon F)_X \\ &= (G(\varepsilon \circ FG\varepsilon)F)_X \\ &= (G(\varepsilon \circ \varepsilon FG)F)_X \\ &= (G\varepsilon F \circ G\varepsilon FGF)_X \\ &= (\mu \circ T\mu)_X.\end{aligned}$$

Also we have,

$$\begin{aligned}(\mu \circ T\eta)_X &= (G\varepsilon F \circ GF\eta)_X \\ &= G(\varepsilon F \circ F\eta)_X \\ &= G(id_F)_X, && \text{by definition 5.7} \\ &= G(id_{F_X}) \\ &= G(F(X)) \\ &= id_{GF_X} \\ &= id_{T_X}\end{aligned}$$

and,

$$\begin{aligned}(\mu \circ \eta T)_X &= (G\varepsilon F \circ \eta GF)_X \\ &= ((G\varepsilon \circ \eta G)F)_X \\ &= ((id_G)F)_X, && \text{by Definition 5.7} \\ &= (id_{G_{F(X)}}) \\ &= id_{GF_X} \\ &= id_{T_X}.\end{aligned}$$

□

Remark 7.14. The above shows that every adjoint situation gives rise to a monad, the following discussion shows that every monad has at least one adjoint situation which gives rise to it.

The following Lemma 7.15 is stated in Adámek - Herrlick - Strecker [1] (Page 319, Proposition 20.7) but not proven.

Lemma 7.15. Let $\mathscr{D}$ be a category and $\mathbf{T} = (T, \eta, \mu)$ be a monad over $\mathscr{D}$. Then $(F^{\mathbf{T}}, G^{\mathbf{T}}, \eta^{\mathbf{T}}, \varepsilon^{\mathbf{T}})$ is an adjoint situation where:

1.
$$\mathrm{G}^{\mathbf{T}} \colon \mathrm{Alg}(\mathbf{T}) \to \mathscr{D}$$
is the *forgetful functor* which sends $(X, \theta_1) \in \mathrm{Ob}(\mathrm{Alg}(\mathbf{T}))$,
$$(X, \theta_1) \mapsto X$$
and $f \in \hom_{\mathrm{Alg}(\mathbf{T})}((X, \theta_1), (Y, \theta_2))$,
$$f \mapsto f$$
since $f \in \hom_{\mathscr{C}}(X, Y)$;

2.
$$\mathrm{F}^{\mathbf{T}} \colon \mathscr{D} \to \mathrm{Alg}(\mathbf{T})$$
sends objects $X \in \mathrm{Ob}(\mathscr{D})$,
$$X \mapsto (T(X), \mu_X)$$
and morphisms $f \in \hom_{\mathscr{D}}(X, Y)$,
$$f \mapsto T(f)$$
where $T(f) \in \hom_{\mathrm{Alg}(\mathbf{T})}((T(X), \mu_X), (T(Y), \mu_Y))$;

3. $$\varepsilon^{\mathbf{T}} \colon F^{\mathbf{T}} G^{\mathbf{T}} \Rightarrow id_{\mathrm{Alg}(\mathbf{T})},$$
   is defined for each $(X, \theta) \in \mathrm{Ob}(\mathrm{Alg}(\mathbf{T}))$ as,
   $$\varepsilon^{\mathbf{T}}(X, \theta) = \theta;$$

4. $\eta^{\mathbf{T}} = \eta,$.

Proof. For each $X \in \mathrm{Ob}(\mathscr{D})$ we have $(X, \mu_X) \in \mathrm{Ob}(\mathrm{Alg}(\mathbf{T}))$ by Lemma 7.8. Then we check $G^{\mathbf{T}}$ and $F^{\mathbf{T}}$ are indeed functors by checking the axioms in Definition 3.1. For any two morphisms, $f \in \hom_{\mathrm{Alg}(\mathbf{T})}((X, \theta_1), (Y, \theta_2))$ and $g \in \hom_{\mathrm{Alg}(\mathbf{T})}((Y, \theta_2), (Z, \theta_3))$ we have,

$$\begin{aligned} G^{\mathbf{T}}(g \circ f) &= g \circ f \\ &= G^{\mathbf{T}}(g) \circ G^{\mathbf{T}}(f). \end{aligned}$$

Given $id_X \in \hom_{\mathrm{Alg}(\mathbf{T})}((X, \theta_1), (X, \theta_1))$ we have,

$$G^{\mathbf{T}}(id_X) = id_X.$$

Therefore $G^{\mathbf{T}}$ is a functor.

Given $f \in \hom_{\mathscr{D}}(X, Y)$ and $g \in \hom_{\mathscr{D}}(Y, Z)$ we have,

$$\begin{aligned} F^{\mathbf{T}}(g \circ f) &= T(g \circ f) \\ &= T(g) \circ T(f) \\ &= F^{\mathbf{T}}(g) \circ F^{\mathbf{T}}(f). \end{aligned}$$

Given $id_X \in \hom_{\mathscr{D}}(X, X)$ we have,

$$\begin{aligned} F^{\mathbf{T}}(id_X) &= T(id_X) \\ &= id_X && \text{since } T \text{ is an endofunctor.} \end{aligned}$$

Therefore $F^{\mathbf{T}}$ is a functor.

η is defined as a natural transformation so we just check the naturality condition on ε. Given $(X, \theta_1), (Y, \theta_2) \in \mathrm{Ob}(\mathrm{Alg}(\mathbf{T}))$ and $f \in \hom_{\mathrm{Alg}(\mathbf{T})}((X, \theta_1), (Y, \theta_2))$we have,

$$\begin{aligned} \varepsilon_{(Y,\theta_2)} \circ F^{\mathbf{T}}(G^{\mathbf{T}}(f)) &= \theta_2 \circ F^{\mathbf{T}}(f) \\ &= \theta_2 \circ T(f) \\ &= f \circ \theta_1, && \text{By Definition 7.6} \\ &= f \circ \varepsilon_{(X,\theta_1)}. \end{aligned}$$

Therefore ε is a natural transformation.

To prove the above defines an adjoint situation we check the axioms in Definition 5.7. For each $X \in \mathrm{Ob}(\mathscr{D})$ we have,

$$\begin{aligned} (\varepsilon F^{\mathbf{T}} \circ F^{\mathbf{T}} \eta)_X &= \varepsilon_{F^{\mathbf{T}}(X)} \circ F^{\mathbf{T}}(\eta_X) \\ &= \varepsilon_{(T(X), \mu_X)} \circ F^{\mathbf{T}}(\eta_X) \\ &= \mu_X \circ F^{\mathbf{T}}(\eta_X) \\ &= (\mu \circ F^{\mathbf{T}} \eta)_X \\ &= id_{F^{\mathbf{T}}} \end{aligned}$$

since $\mathbf{T}$ is a monad and $F^{\mathbf{T}}$ and there is a one to one correspondence between $F^{\mathbf{T}}(X)$ and $T(X)$. For each $(X, \theta) \in \mathrm{Ob}(\mathrm{Alg}(\mathbf{T}))$ we have,

$$\begin{aligned}(G^{\mathbf{T}}\varepsilon \circ \eta G^{\mathbf{T}})_{(X,\theta)} &= G^{\mathbf{T}}(\varepsilon_{(X,\theta)}) \circ \eta_{G^{\mathbf{T}}((X,\theta))} \\ &= G^{\mathbf{T}}(\theta) \circ \eta_X \\ &= \theta \circ \eta_X \\ &= id_X, && \text{since } (X,\theta) \text{ is an algebra of } \mathbf{T} \\ &= id_{G^{\mathbf{T}}((X,\theta))}.\end{aligned}$$

Therefore $(F^{\mathbf{T}}, G^{\mathbf{T}}, \eta, \varepsilon)$ is an adjoint situation. □

Remark 7.16. The above Lemma 7.15 gives only existence of an adjoint situation given a monad and not uniqueness. In general a monad will *not* give rise to a unique adjoint situation.

We now define an important subcategory of the Eilenberg-Moore category of a monad, $\mathbf{T}$, the Kleisli category, and see how an adjoint situation from this category also gives rise to the monad $\mathbf{T}$. See Wiki [16] for more details on the Kleisli category.

Definition 7.17. Let $\mathscr{C}$ be a category and $\mathbf{T} = (T, \eta, \mu)$ be a monad over $\mathscr{C}$ then we define the *Kleisli category*, $K_{\mathbf{T}}$ as the full subcategory of $\mathrm{Alg}(\mathbf{T})$ whose objects are the *free objects*, $(T(X), \eta_X)$ for each $X \in \mathrm{Ob}(\mathscr{C})$.

The following lemma is adapted from MacLane [6] (Page 148, Theorem 3)

Definition 7.18. Let $\mathscr{C}$ be a category and $\mathbf{T} = (T, \eta, \mu)$ be a monad over $\mathscr{C}$. We define *the category of adjoint situations for the monad* $\mathbf{T}$, $adj_{\mathbf{T}}$ as:

1. The objects, $(\mathrm{F}\colon \mathscr{C} \to \mathscr{X}, \mathrm{G}\colon \mathscr{X} \to \mathscr{C}, \eta, \varepsilon) \in \mathrm{Ob}(adj_{\mathbf{T}})$, are adjoint situations which give rise to the monad $\mathbf{T}$, that is $(GF, \eta, G\varepsilon F) = (T, \eta, \mu)$;
2. The morphisms, $(K, id_{\mathscr{X}})$ are morphisms of adjoint situations defined in Definition 5.12 such that $id_{\mathscr{X}}$ is the identity on $\mathscr{X}$;
3. Composition is composition of morphisms of adjoint situations as defined in Definition 5.13;
4. The identities are defined in Example 5.15.

Lemma 7.19. $adj_{\mathbf{T}}$ as defined above in Definition 7.18 is a category.

Proof. The composition is associative since the composition of functors is associative. Hence we have a category structure. □

The following Lemma 7.20 can be found in MacLane [6] with no proof, here we omit the proof.

Lemma 7.20. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad and $adj_{\mathbf{T}}$ be the corresponding category of adjoint situations. Then the terminal object in $adj_{\mathbf{T}}$ is the adjoint situation, $(\mathrm{F}^{\mathbf{T}}\colon \mathscr{D} \to \mathrm{Alg}(\mathbf{T}), \mathrm{G}^{\mathbf{T}}\colon \mathrm{Alg}(\mathbf{T}) \to \mathscr{D}, \eta, \varepsilon)$, defined in Lemma 7.15.

This result ties together different parts of this project. Had this project been longer we would have discussed the proof and consequences. We also could have looked into monoidal categories and how monoids generalise in other categories, this leads to the exploration of the quote by James Iry [4]: "a monad is a monoid in the category of endofunctors, what's the problem?".

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