# Symmetries of M.C. Escher's Paintings

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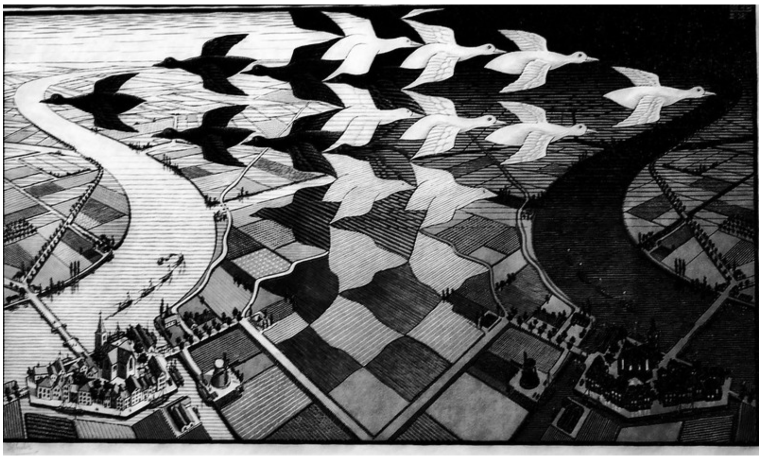

Figure 1: Escher (1938)

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# 1 Introduction

Figure 1 was the first image which drew me to studying this project; clearly it's a beautiful piece of artwork but it also seems to contain a fundamental 'amount' of symmetry. It is known that the study of group theory is often thought as the study of symmetry but its not always easy to find the explicit link between the two. This report will detail how we can apply knowledge of group theory to categorise fundamental symmetries of 2-Dimensional patterns such as the one seen in Figure 1, known as wallpaper patterns.

## 1.1 Ethics

Its important to think about the ethics of this kind of research and how it will effect peoples lives. This area of mathematics is often called crystallography as if we extend to a 3 dimensional euclidean space it can be applied to the study of crystal structures created by atoms. This is important ethically since if we can understand the structure of material it could help us develop technologies which could have a beneficial impact on human lives. This could for example help us develop a new technology to treat wounds which could in turn help save lives.

This subject is also worthwhile to research because it is develops fundamental theories of mathematics such as Group theory which can be applied to many different areas of science. Group theory is a vast topic which in essence helps us understand fundamental symmetries. If we understand symmetries of problems this often helps us solve them which in turn would help the speed of other research which may increase the quality of peoples lives.

# 2 Isometries and the Euclidean group

## 2.1 Motivation

To motivate the mathematics behind these patterns we need to think about how to act on the plane $\mathbb{R}^2$ while preserving some structure, in fact we try to think how to act on the plane while preserving *distance between points*. We first notice patterns seem to be preserved when reflected, translated or rotated, these are collectively known as isometries of the plane which leads us to the following definition.

## 2.2 Isometries

These following sections contain ideas and notions from Armstrong (1988). We start by defining an isometry.

**Definition 2.1.** A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** provided it preserves distance between points, that is:

$$\left\| f(\underline{\mathbf{x}}) - f(\underline{\mathbf{y}}) \right\| = \left\| \underline{\mathbf{x}} - \underline{\mathbf{y}} \right\| \tag{2.1}$$

for every pair of points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

This definition makes intuitive sense because we want to move the pattern without distorting how it looks. In our case we restrict to looking at isometries in $\mathbb{R}^2$ only.

We will find that the set of all isometries forms a group under the group law of function composition called the **Euclidean Group**, $E_2$. To prove this note the identity element for functions, $id$, is an isometry since for each pair of points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$

$$\left\| id(\underline{\mathbf{x}}) - id(\underline{\mathbf{y}}) \right\| = \left\| \underline{\mathbf{x}} - \underline{\mathbf{y}} \right\| \tag{2.2}$$

and for any two isometries $f, g \in E_2$ we have,

$$\begin{aligned} \left\| f(g(\underline{\mathbf{x}})) - f(g(\underline{\mathbf{y}})) \right\| &= \left\| g(\underline{\mathbf{x}}) - g(\underline{\mathbf{y}}) \right\| && \textit{def } 2.1 \\ &= \left\| (\underline{\mathbf{x}} - \underline{\mathbf{y}}) \right\| && \textit{def } 2.1 \end{aligned}$$

Thus $f \circ g \in E_2$. Furthermore each isometry $f$ is a bijection so has an inverse $f^{-1}$ which satisfies,

$$\begin{aligned} \left\| f^{-1}(\underline{\mathbf{x}}) - f^{-1}(\underline{\mathbf{y}}) \right\| &= \left\| f(f^{-1}(\underline{\mathbf{x}})) - f(f^{-1}(\underline{\mathbf{y}})) \right\| && \textit{since f is an isometry} \\ &= \left\| \underline{\mathbf{x}} - \underline{\mathbf{y}} \right\| \end{aligned}$$

Therefore $f^{-1} \in E_2$. Since we know function composition is associative we can conclude $E_2$ is indeed a group. Our goal for this section is to recognise the Euclidean group in a form we can easily manipulate and understand. Less formally we know isometries as rotations, reflections and translations. Note a translation by $\underline{\mathbf{v}}$ is an isometry $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\tau(\underline{\mathbf{x}}) = \underline{\mathbf{v}} + \underline{\mathbf{x}}$ for some vector $\underline{\mathbf{v}} \in \mathbb{R}^2$, therefore $\tau(\underline{\mathbf{0}}) = \underline{\mathbf{v}}$.

We can show now that every isometry can be written as a either a rotation about the origin followed by a translation, or a reflection by a line through the origin followed by a translation. Suppose $g \in E_2$ is an isometry which sends the origin to point $\underline{\mathbf{v}}$ ,let $\tau \in E_2$ be a translation such that $\tau(\underline{\mathbf{x}}) = \underline{\mathbf{v}} + \underline{\mathbf{x}}$. Now we can define the isometry $f = \tau^{-1}g$ which fixes the origin. We claim that f will be either a rotation about the origin or a reflection by a line through the origin. We introduce a method to reference each point on the plane by taking $\underline{\mathbf{p}} = (1, 0)$ and $\underline{\mathbf{q}} = (0, 1)$ then each point $\underline{\mathbf{x}} \in \mathbb{R}^2$ is uniquely determined by the three measurements:

$$\|\underline{\mathbf{x}}\|, \quad \left\|\underline{\mathbf{x}} - \underline{\mathbf{p}}\right\|, \quad \left\|\underline{\mathbf{x}} - \underline{\mathbf{q}}\right\|$$

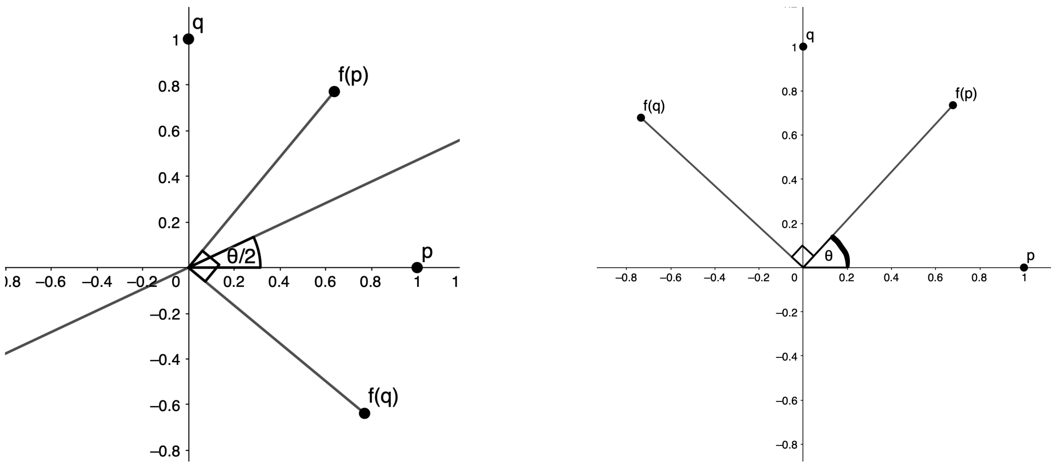
However, since f is an isometry,

$$\|f(\underline{\mathbf{x}})\| = \|\underline{\mathbf{x}}\|, \quad \left\|f(\underline{\mathbf{x}}) - f(\underline{\mathbf{p}})\right\| = \left\|\underline{\mathbf{x}} - \underline{\mathbf{p}}\right\|, \quad \left\|f(\underline{\mathbf{x}}) - f(\underline{\mathbf{q}})\right\| = \left\|\underline{\mathbf{x}} - \underline{\mathbf{q}}\right\|$$

So if we know the position of $f(\underline{\mathbf{p}})$ and $f(\underline{\mathbf{q}})$ in the plane, then we know where f sends every point in the plane. We also know,

$$\left\|f(\underline{\mathbf{p}})\right\| = \left\|\underline{\mathbf{p}}\right\| = 1, \quad \left\|f(\underline{\mathbf{q}})\right\| = \left\|\underline{\mathbf{q}}\right\| = 1, \quad \left\|f(\underline{\mathbf{p}}) - f(\underline{\mathbf{q}})\right\| = \left\|\underline{\mathbf{p}} - \underline{\mathbf{q}}\right\| = \sqrt{2}$$

and therefore the angle $\angle f(\underline{\mathbf{p}})\underline{\mathbf{0}}f(\underline{\mathbf{q}})$ is a right angle. So the image of $\underline{\mathbf{p}}$ must stay on the unit circle, we can reference this by an anticlockwise rotation through $\theta$, we can therefore use trigonometry to deduce the remaining possible positions for the image of $\underline{\mathbf{q}}$. If $\underline{\mathbf{p}}$ rotates to $f(\underline{\mathbf{p}})$ then $\underline{\mathbf{q}}$ has only two options which remain perpendicular, see figure 2. In the first case $f$ is an anticlockwise rotation through $\theta$ about the origin, figure 2b, otherwise $f$ is a reflection in a line through the origin, angle $\frac{\theta}{2}$ with the positive x-axis, figure 2a. Since $g$ is a general isometry and is $g = \tau f$ then every isometry can be written as a rotation or reflection followed by a translation.




(a) A reflection (b) A rotation

Figure 2

To show the translations are a subgroup of the Euclidean group we can apply a theorem from group theory which states:

**Theorem 2.2.** *A non-empty subset H of a group G is a subgroup if and only if* $xy^{-1} \in H$ *whenever* $x, y \in H$.

This is a standard result from group theory so the proof will be omitted but can be found in Armstrong (1988) Ch.5 .So for any translation $\tau_1, \tau_2 \in T$ where $\tau_1(\underline{\mathbf{x}}) = \underline{\mathbf{v}} + \underline{\mathbf{x}}$ and $\tau_2(\underline{\mathbf{x}}) = \underline{\mathbf{u}} + \underline{\mathbf{x}}$, $\forall \underline{\mathbf{x}} \in \mathbb{R}^2$

$$\begin{aligned} \tau_1 \tau_2^{-1}(\underline{\mathbf{x}}) &= \tau_1(-\underline{\mathbf{u}} + \underline{\mathbf{x}}) \\ &= -\underline{\mathbf{u}} + (\underline{\mathbf{v}} + \underline{\mathbf{x}}) \\ &= (\underline{\mathbf{v}} - \underline{\mathbf{u}}) + \underline{\mathbf{x}} \end{aligned}$$

This is a translation by $\underline{\mathbf{v}} - \underline{\mathbf{u}}$ therefore the translations form a subgroup of the Euclidean group. In a similar way we can show the group of orthogonal transformations is a subgroup, $O \leq E_2$ this is the group of a rotations about the origin and reflections by lines through the origin. So above we have shown $E_2 = TO$.

We know $T \cap O = \{id\}$ since every translation moves the origin other than the identity and every orthogonal transformation fixes the origin. We can now show a fundamental result that each isometry can be written *uniquely* as an orthogonal transformation followed by a translation. Suppose $g \in E_2$ can be written in two ways, $g = \tau f = \tau' f'$ with $\tau, \tau' \in T$ and $f, f' \in O$ then $(\tau')^{-1}\tau = (f')^{-1} f = id$ since this lies in the intersection $T \cap O$ therefore $\tau = \tau'$ and $f = f'$ and so $g = \tau f$ is unique. Let $g = \tau f$ where $\tau \in T$ then if $f$ is a rotation we call $g$ a ***direct isometry*** and if $f$ is a reflection then we call $g$ a ***opposite isometry***.

Another piece of the puzzle is to show the translation subgroup is normal,

**Definition 2.3.** Let G be a group, a subgroup $H \leq G$ is said to be normal if $\forall g \in G$ and $h \in H$, $ghg^{-1} \in H$, written $H \trianglelefteq G$.

T is a subgroup so $(\tau')\tau(\tau')^{-1} \in T$ for all $\tau, \tau' \in T$. So we need to show $f\tau f^{-1} \in T$, for all $f \in O$ and $\tau \in T$. For each $\underline{\mathbf{x}} \in \mathbb{R}^2$,

$$\begin{aligned} f\tau f^{-1}(\underline{\mathbf{x}}) &= f(\underline{\mathbf{v}} + f^{-1}(\underline{\mathbf{x}})) \\ &= f(\underline{\mathbf{v}}) + f(f^{-1}(\underline{\mathbf{x}})) && \textit{since f is linear} \\ &= f(\underline{\mathbf{v}}) + \underline{\mathbf{x}} \end{aligned}$$

Thus $f\tau f^{-1}$ is a translation by vector $f(\underline{\mathbf{v}})$. Therefore $T$ is a normal subgroup of $E_2$

### 2.3 The semidirect product

This subsection will use ideas from chapter 23 of Armstrong (1988) which talks about automorphisms between groups. Now we can find how the product of two general elements of $E_2$ works in terms of the translation and orthogonal subgroups. So given two elements, $g, h \in E_2$, we know $g = \tau f$ and $h = \tau_1 f_1$ for $\tau, \tau_1 \in T$ and $f, f_1 \in O$. So,

$$\begin{aligned} gh &= \tau f \tau_1 f_1 \\ &= \tau f \tau_1 (f^{-1} f) f_1 && \textit{since } f^{-1}f = id \\ &= (\tau f \tau_1 f^{-1})(f f_1) \end{aligned}$$

Hence $gh$ is the orthogonal transformation $ff_1$ followed by the translation $\tau f \tau_1 f^{-1}$. We state this formally using the following definitions.

**Definition 2.4.** Let G be a group, an **automorphism of G** is an isomorphism from G to G. The set of all automorphisms of G forms a group under function composition called the **automorphism group of G** written Aut(G).

For example the isomorphism $\phi_h : G \to G$, where for a given $h \in G$ and all $g \in G$, $\phi_h(g) \mapsto hgh^{-1}$, is an automorphism of G. That is conjugation by an element $h \in G$ is an automorphism of G.

**Definition 2.5.** Let $(G, \bullet)$ and $(H, *)$ be groups and $\phi : H \to Aut(G)$ be a homomorphism between groups where $\phi(h) \mapsto \phi_h$. The **semidirect product** of G and H determined by $\phi$ is defined $G \times_\phi H$. It's elements are ordered pairs $(g, h)$ where $g \in G$ and $h \in H$ and the multiplication is defined,

$$(g, h)(g', h') = (g \bullet \phi_h(g'), h * h')$$

where $\bullet$ is the multiplication of G and $*$ is the multiplication of H.

The proof that the semidirect product is indeed a group will be left in Appendix A. This may seem overwhelming at first but note in this definition $\phi_h$ is just an automorphism of G. Continuing the above example $\phi_h(g') = hg'h^{-1}$, thus the first part of the pair would be determined by multiplying $hg'h^{-1}$ on the left by $g$, i.e. $ghg'h^{-1}$. Furthermore, for our purposes $\bullet$ and $*$ may necessarily be the same. Finally we need a theorem from group theory,

**Theorem 2.6.** *Let E be a group and G,H subgroups of E. If G is a normal subgroup, if GH=E and $G \cap H = \{e\}$ where e is the identity of E, then E is isomorphic to the semidirect product $G \times_\phi H$, where $\phi : H \to Aut(G)$ is the homomorphism defined $\phi(h)(g) \mapsto \phi_h(g) \mapsto hgh^{-1}$ for all $h \in H$ and $g \in G$.*

The proof of this theorem can be found in Appendix A. We now have the tools to show how each element $g$ of the Euclidean group can be written as an ordered pair $(\tau, f)$ with $\tau \in T$ and $f \in O$ and state what the multiplication looks like.

**Corollary 2.6.1.** *For the Euclidean group $E_2$ with translation and orthogonal subgroups T and O respectively, the homomorphism:*

$$\begin{gathered} E_2 \to T \times_\phi O \\ g \mapsto (\tau, f) \end{gathered}$$

*where $T \times_\phi O$ is the semidirect product of T and O where $\phi : O \to Aut(T)$ is given by conjugation, $\phi(f)(\tau) \mapsto \phi_f(\tau) \mapsto f\tau f^{-1}$ for all $\tau \in T$, is an* ***isomorphism****.*

Clearly this result follows directly from Theorem 2.6 as $O \leq E_2$ and $T \trianglelefteq E_2$. This is a good way of thinking about the Euclidean group but for calculations we can use a more useful equivalence of the Euclidean group. Let $g = \tau f$ be a general isometry, if $\tau$ is given by $\tau(\underline{\mathbf{x}}) = \underline{\mathbf{v}}+\underline{\mathbf{x}}$ and $M$ is the orthogonal matrix which represents the orthogonal transformation $f$ in the standard basis for $\mathbb{R}^2$, we have for a general point $\underline{\mathbf{x}} = (x, y) \in \mathbb{R}^2$

$$g(\underline{\mathbf{x}}) = \underline{\mathbf{v}} + f_M(\underline{\mathbf{x}}) = \underline{\mathbf{v}} + \underline{\mathbf{x}}M^t \tag{2.3}$$

conversely given a vector $\underline{\mathbf{v}} \in \mathbb{R}^2$ and a matrix $M \in O_2$ ($O_2$ is the set of orthogonal transformation matrices in $\mathbb{R}^2$), we have a *unique* isometry of the plane. So we can write each isometry as an ordered pair $(\underline{\mathbf{v}}, M)$ with multiplication given by

$$(\underline{\mathbf{v}}, M)(\underline{\mathbf{v}}_1, M_1) = (\underline{\mathbf{v}} + f_M(\underline{\mathbf{v}}_1), MM_1) \tag{2.4}$$

this again is a way of writing $E_2$ as a semidirect product,

**Corollary 2.6.2.** *For the Euclidean group $E_2$ the homomorphism:*

$$\begin{aligned} E_2 &\to \mathbb{R}^2 \times_\psi O_2 \\ g &\mapsto (\underline{\mathbf{v}}, M) \end{aligned}$$

*where $\mathbb{R}^2 \times_\psi O_2$ is the semidirect product of $\mathbb{R}^2$ and $O_2$ where $\psi : O_2 \to Aut(\mathbb{R}^2)$ is given by the usual action of $O_2$ on $\mathbb{R}^2$, $\psi(M)(\underline{\mathbf{v}}) \mapsto \psi_M(\underline{\mathbf{v}}) \mapsto \underline{\mathbf{v}}M^t$ for all $\tau \in T$, is an* ***isomorphism****.*

This is easy to see since $T \cong \mathbb{R}^2$ and $O \cong O_2$ and

$$\begin{aligned} f_M \tau f_M^{-1}(\underline{\mathbf{x}}) &= f_M(\underline{\mathbf{v}} + f_M^{-1}(\underline{\mathbf{x}})) \\ &= f_M(\underline{\mathbf{v}}) + f_M(f_M^{-1}(\underline{\mathbf{x}})) \\ &= f_M(\underline{\mathbf{v}}) + \underline{\mathbf{x}} \end{aligned}$$

So we have found a useful way to write elements of the Euclidean group, as an ordered pair of a vector representing a translation and a matrix representing an orthogonal transformation.

## 2.4 Categorising the isometries

We will now look at how to categorise different types of isometries by looking at the ordered pair $(\underline{\mathbf{v}}, M)$. There are two options for the matrix $M \in O_2$: a rotation matrix $A$ or a reflection matrix $B$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} \tag{2.5}$$

Where $\theta$ is the angle of anticlockwise rotation and $\frac{\phi}{2}$ is the angle the line of reflection makes with the x-axis. Notice $A$ represents a **direct isometry** with $\det A = +1$ and $B$ represents an **opposite isometry** with $\det B = -1$.

We consider 5 different cases, firstly:

**(a)** A translation by the vector $\underline{\mathbf{v}}$ is written as an ordered pair $(\underline{\mathbf{v}}, I)$. this is straight forward as to have a pure translation we need no rotation or reflection so the identity matrix is the only option.

**(b)** A rotation anticlockwise through $\theta$ about the origin is written as an ordered pair $(\underline{\mathbf{0}}, A)$. This is since we need no translation to rotate about the origin since that is what the matrix

A does.
**(c)** A rotation anticlockwise through $\theta$ about the point $\underline{\mathbf{c}}$ is written as an ordered pair $(\underline{\mathbf{c}} - f_A(\underline{\mathbf{c}}), A)$. To rotate about $\underline{\mathbf{c}}$ we first need to translate by $-\underline{\mathbf{c}}$ to move $\underline{\mathbf{c}}$ to the origin, then we rotate about the origin using the matrix A and then translate back to $\underline{\mathbf{c}}$, so:

$$\begin{aligned}\underline{\mathbf{c}} + f_A(\underline{\mathbf{x}} - \underline{\mathbf{c}}) &= \underline{\mathbf{c}} + f_A(\underline{\mathbf{x}}) - f_A(\underline{\mathbf{c}}) \\ &= (\underline{\mathbf{c}} - f_A(\underline{\mathbf{c}})) + f_A(\underline{\mathbf{x}})\end{aligned}$$

**(d)** A reflection in the line l (see figure 3) is written as an ordered pair $(\underline{\mathbf{0}}, B)$. Again no translation is needed since the matrix B already reflects in the line l.
**(e)** A reflection in the line m is written as an ordered pair $(2\underline{\mathbf{a}}, B)$. Here we need to translate by $-\underline{\mathbf{a}}$ to move the line m to l, reflect in the origin then translate back by $\underline{\mathbf{a}}$ so for a general vector $\underline{\mathbf{x}} \in \mathbb{R}$:

$$\begin{aligned}\underline{\mathbf{a}} + f_B(\underline{\mathbf{x}} - \underline{\mathbf{a}}) &= \underline{\mathbf{a}} + f_B(\underline{\mathbf{x}}) - f_B(\underline{\mathbf{a}}) \\ &= \underline{\mathbf{a}} + f_B(\underline{\mathbf{x}}) + \underline{\mathbf{a}} && \textit{since } f_B(\underline{\mathbf{a}}) = -\underline{\mathbf{a}} \\ &= 2\underline{\mathbf{a}} + f_B(\underline{\mathbf{x}})\end{aligned}$$

**(f)** If we have a reflection followed by a translation parallel to the line of reflection we call it a **glide reflection**. If we take m as the line of reflection then a glide reflection along m is written as an ordered pair $(2\underline{\mathbf{a}} + \underline{\mathbf{b}}, B)$ where $f_B(\underline{\mathbf{b}}) = \underline{\mathbf{b}}$ and $\underline{\mathbf{b}} \neq \underline{\mathbf{0}}$

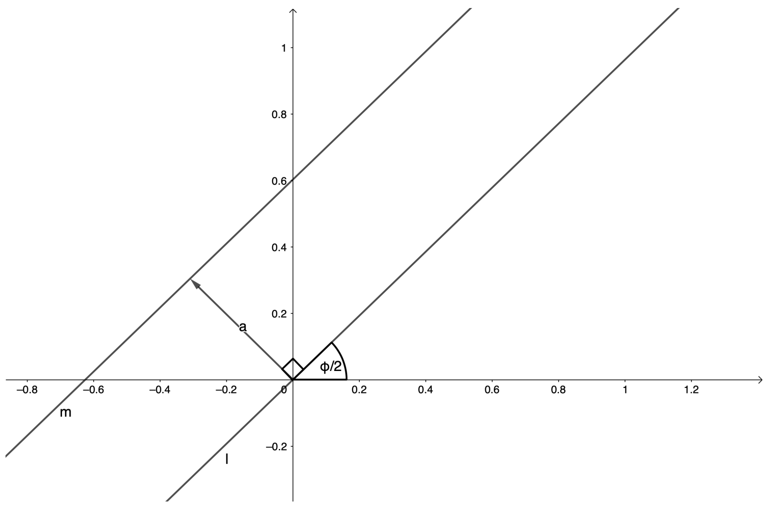



Figure 3

This leads us to a crucial theorem about every isometry,

**Theorem 2.7.** *Every direct isometry is a translation or a rotation. Every opposite isometry is a reflection or a glide reflection.*

*Proof.* A direct isometry is written as an ordered pair, $(\underline{\mathbf{v}}, A)$ with A as in 2.5 and $0 \leq \theta < 2\pi$. If $\theta = 0$ then the pair is $(\underline{\mathbf{v}}, I)$, a translation by $\underline{\mathbf{v}}$. If $\theta \neq 0$ then the isometry is a rotation about the point $f_{I-A}^{-1}(\underline{\mathbf{v}})$ since,

$$\det(I - A) = \det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 2 - 2\cos\theta > 0 \tag{2.6}$$

for all $\theta$. So $I - A$ is invertible and the equation,

$$\underline{\mathbf{c}} - f_A(\underline{\mathbf{c}}) = f_{I-A}(\underline{\mathbf{c}}) = \underline{\mathbf{v}} \tag{2.7}$$

has a unique solution, $\underline{\mathbf{c}} = f_{I-A}^{-1}(\underline{\mathbf{v}})$. Hence we have case **(c)** above, a rotation. An opposite isometry is written as an ordered pair, $(\underline{\mathbf{v}}, B)$ with B as in 2.5 and $0 \leq \phi < 2\pi$. If $f_B(\underline{\mathbf{v}}) = -\underline{\mathbf{v}}$ then we have case **(e)** above, a reflection in the line m where $\underline{\mathbf{a}} = \frac{\mathbf{v}}{2}$. If $f_B(\underline{\mathbf{v}}) \neq -\underline{\mathbf{v}}$ set $\underline{\mathbf{w}} = \underline{\mathbf{v}} - f_B(\underline{\mathbf{v}})$ and then,

$$\begin{aligned} f_B(\underline{\mathbf{w}}) &= f_B(\underline{\mathbf{v}} - f_B(\underline{\mathbf{v}})) \\ &= f_B(\underline{\mathbf{v}}) - f_B^2(\underline{\mathbf{v}}) \\ &= f_B(\underline{\mathbf{v}}) - \underline{\mathbf{v}} \\ &= -\underline{\mathbf{w}} \end{aligned}$$

So we can resolve $\underline{\mathbf{v}}$ along $\underline{\mathbf{w}}$ to get the vector $(\underline{\mathbf{v}}.\underline{\mathbf{w}}/\left\|\underline{\mathbf{w}}\right\|^2)\underline{\mathbf{w}}$. This represents a glide reflection as in case **(f)** with $2\underline{\mathbf{a}} = (\underline{\mathbf{v}}.\underline{\mathbf{w}}/\left\|\underline{\mathbf{w}}\right\|^2)\underline{\mathbf{w}}$ and $\underline{\mathbf{b}} = \underline{\mathbf{v}} - 2\underline{\mathbf{a}}$. □

We now know how every isometry behaves and have them in a form we can use for calculations, next we will look at how these isometries act on the plane but first some examples.

## 2.5 Examples

**Example 2.5.1.** Suppose we have the isometry f, an anticlockwise rotation about the point $(-1, 1)$ and we want to write as an ordered pair $(\underline{\mathbf{v}}, M)$. We see in section 2.4 that this will fall into case **(c)** and the pair will look like $(\underline{\mathbf{c}} - f_A(\underline{\mathbf{c}}), A)$ with $\underline{\mathbf{c}}$ the centre of rotation and A as in 2.5. So in this case,

$$\begin{aligned} A &= \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \end{aligned}$$

Then,

$$\begin{aligned} \underline{\mathbf{c}} - f_A(\underline{\mathbf{c}}) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 + \sqrt{3} \\ 3 - \sqrt{3} \end{bmatrix} \end{aligned}$$

So written as an ordered pair this isometry is $(\underline{\mathbf{v}}, A)$ with $\underline{\mathbf{v}} = \frac{1}{2}(-1 + \sqrt{3}, 3 - \sqrt{3})$ and A as above.

**Example 2.5.2.** Suppose we have a reflection in the line $x + y + 3 = 0$ and we want to write it as an ordered pair, $(\underline{\mathbf{v}}, M)$. This aligns with case **(e)** in section 2.4 where the pair can be written such as $(2\underline{\mathbf{a}}, B)$ where $\underline{\mathbf{a}}$ is the perpendicular vector from the origin to the line and B is the matrix in 2.5 where $\epsilon = \frac{\phi}{2}$ is the angle the line makes with the x-axis. We draw a diagram,

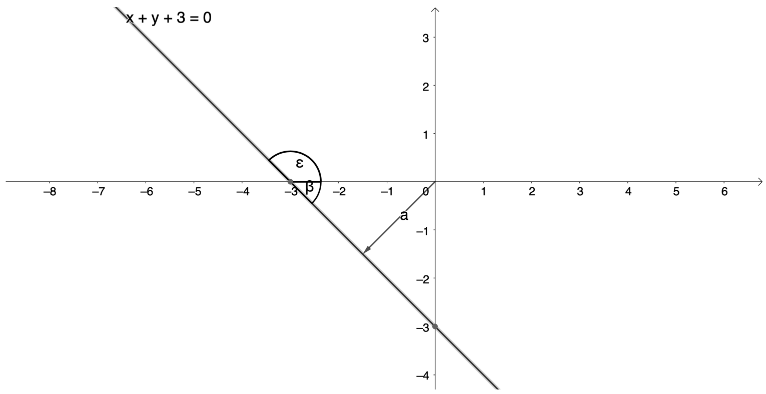



Figure 4

We can use simple trigonometry to deduce, $\epsilon = \frac{3\pi}{4}$ hence, $\phi = \frac{3\pi}{2}$. Therefore the matrix B is given,

$$\begin{aligned} B &= \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{3\pi}{2} & \sin\frac{3\pi}{2} \\ \sin\frac{3\pi}{2} & -\cos\frac{3\pi}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

One way of finding $\underline{\mathbf{a}} = (a, b)$ is noting $\underline{\mathbf{a}} \cdot (1, -1) = 0$ since $(1, -1)$ is a vector on the line and $\underline{\mathbf{a}} = (a, b)$ is perpendicular to the line then,

$$\underline{\mathbf{a}} \cdot (1, -1) = a - b$$

$$\begin{aligned} &\implies a = b \\ &\implies \underline{\mathbf{a}} = (a, a) \quad a \in \mathbb{R} \end{aligned}$$

then since $(a, a)$ must be on the line we plug back into the equation to get,

$$\begin{aligned} a + a + 3 = 0 \\ \implies a = -\frac{3}{2} \end{aligned}$$

and so $2\underline{\mathbf{a}} = (-3, -3)$.

**Example 2.5.3.** We will show that a reflection in a line m followed by a reflection in the line m′ is a translation when m is parallel to m′ and a rotation otherwise. If g and g′ are reflections in the line m and m′ they can be written as an ordered pair,

$$g = (2\underline{\mathbf{a}}, B), \qquad g' = (2\underline{\mathbf{a}}', B')$$

$$B = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}$$

from case (e) in 2.4. Therefore composing these isometries gives.

$$\begin{aligned} gg' &= (2\underline{\mathbf{a}}, B)(2\underline{\mathbf{a}}', B') \\ &= (2\underline{\mathbf{a}} + f_B(2\underline{\mathbf{a}}'), BB') \qquad \textit{equation } 2.3 \end{aligned}$$

If g and g′ are parallel then they have the same angle $\phi$ with the x axis so $B = B'$ hence $BB' = I$ since $B$ is a reflection matrix. Similarly if g and g′ are parallel then $f_B(2\underline{\mathbf{a}}') = -2\underline{\mathbf{a}}'$ since $\underline{\mathbf{a}}'$ is perpendicular to m. hence,

$$\begin{aligned}(2\underline{\mathbf{a}} + f_B(2\underline{\mathbf{a}}'), BB') &= (2\underline{\mathbf{a}} - 2\underline{\mathbf{a}}', I)\\ &= (2(\underline{\mathbf{a}} - \underline{\mathbf{a}}'), I)\end{aligned}$$

which is a translation by $2(\underline{\mathbf{a}} - \underline{\mathbf{a}}')$. If g and g′ are not parallel then.

$$\begin{aligned}BB' &= \begin{bmatrix}\cos\phi & \sin\phi\\ \sin\phi & -\cos\phi\end{bmatrix}\begin{bmatrix}\cos\phi' & \sin\phi'\\ \sin\phi' & -\cos\phi'\end{bmatrix}\\ &= \begin{bmatrix}\cos\phi\cos\phi' + \sin\phi\sin\phi' & \cos\phi\sin\phi' - \sin\phi\cos\phi'\\ \sin\phi\cos\phi' - \cos\phi\sin\phi' & \sin\phi\sin\phi' + \cos\phi\cos\phi'\end{bmatrix}\\ &= \begin{bmatrix}\cos\phi - \phi' & -\sin\phi - \phi'\\ \sin\phi - \phi' & \cos\phi - \phi'\end{bmatrix} \qquad \textit{using trig summation identities}\end{aligned}$$

This is the matrix A in 2.5 with $\theta = \phi - \phi'$, a rotation matrix. Hence if g and g′ are not parallel their composition is a rotation.

# 3 Wallpaper groups and crystallographic restriction

## 3.1 Wallpaper groups

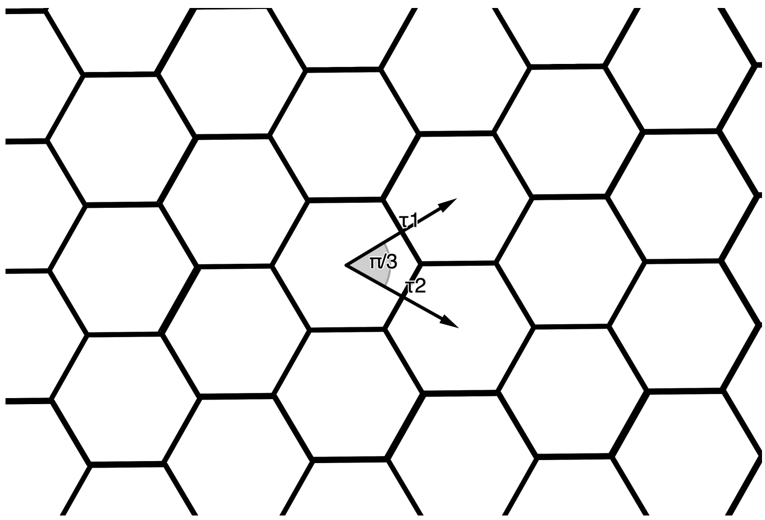



Figure 5: A wallpaper pattern

Ideas in this section will follow ideas in chapter 25 of Armstrong (1988). If we look at figure 5 we see their are only certain isometries which preserve the pattern such as a rotation by $\frac{\pi}{3}$. Again the symmetries which preserve each pattern form a subgroup, $G \leq E_2$ whose elements are the isometries of the plane which send the pattern to itself, called **wallpaper groups**. To correctly define wallpaper groups we think about how their action on the plane is restricted. Using the notation from the previous section we can define a map:

$$\begin{aligned}\pi : E_2 &\to O_2 \\ \pi((\underline{\mathbf{v}}, M)) &\mapsto M\end{aligned} \tag{3.1}$$

This is certainly a homomorphism since,

$$\begin{aligned}\pi((\underline{\mathbf{v}}, M)(\underline{\mathbf{v}}', M')) &= \pi((\underline{\mathbf{v}} + f_M(\underline{\mathbf{v}}'), MM')) \\ &= MM' \\ &= \pi((\underline{\mathbf{v}}, M))\pi((\underline{\mathbf{v}}', M'))\end{aligned}$$

The kernel of this homomorphism is the set of all elements of the form $(\underline{\mathbf{v}}, I)$ since $I$ is the identity of $O_2$, we recognise these as the set of translations. If we restrict the homomorphism to the subgroup G then we call the kernel $H = \ker(\pi_{\restriction G})$ the **translation subgroup** and the image $J = \pi(G)$ the **point group**. Looking at figure 5 we see each patterns translation subgroup can be generated by two independent translations; If H was generated by one translation then there would be no symmetry for points not on that vectors span, if H was generated by 3 or more vectors there could be no pattern since the point $(0, 0)$ could be translated to two other points arbitrarily close together and hence all points must look the same. We may also notice that only a finite number of orthogonal transformations are permissible hence the point group is finite. Both of these observations are stated in Coxeter (1989). This leads us to a rigorous definition.

**Definition 3.1.** A subgroup of $E_2$ is a **wallpaper group** if its translation subgroup is generated by two independent translations and its point group is finite.

## 3.2 The lattice

Now we know what structure these groups should have we think about how they can act on the plane, we will be able to restrict our view to a **lattice**.

**Definition 3.2.** Let G be a wallpaper group with translation subgroup H and point group J. We define the **action of H on** $\mathbb{R}^2$ as:

$$\begin{aligned}&\alpha : H \times \mathbb{R}^2 \to \mathbb{R}^2 \\ &\alpha((\underline{\mathbf{v}}, I), \underline{\mathbf{x}}) \mapsto \underline{\mathbf{x}} + \underline{\mathbf{v}}\end{aligned}$$

We define the set L as the **orbit of the origin under the action**.

$$L = Orb_\alpha(\underline{\mathbf{0}}) = \{\underline{\mathbf{x}} \mid \underline{\mathbf{x}} = \alpha((\underline{\mathbf{v}}, I), \underline{\mathbf{0}})\ \forall\, (\underline{\mathbf{v}}, I) \in H\}$$

This definition may seem confusing at first but it is just all the possible points which can be reached from the only using only the translations in H. L certainly contains two independent vectors since H is generated by two independent translations, we now see exactly what L looks like.

**Theorem 3.3.** *Let* $\underline{\mathbf{a}}$ *be the smallest non-zero vector in L and* $\underline{\mathbf{b}}$ *be the smallest vector linearly independent from* $\underline{\mathbf{a}}$*. The set L is the* ***lattice*** *spanned by* $\underline{\mathbf{a}}$ *and* $\underline{\mathbf{b}}$*. That is,*

$$L = \{m\underline{\mathbf{a}} + n\underline{\mathbf{b}} \mid m, n \in \mathbb{Z}\} \tag{3.2}$$

*Proof.* Consider the map $\alpha_{\underline{\mathbf{0}}}$,

$$\begin{aligned}&\alpha_{\underline{\mathbf{0}}} : T \to \mathbb{R}^2 \\ &(\underline{\mathbf{v}}, I) \mapsto \underline{\mathbf{v}}\end{aligned}$$

Where T is the group of translations. This is a homomorphism since

$$\begin{aligned}\alpha_{\underline{\mathbf{0}}}((\underline{\mathbf{v}}, I)(\underline{\mathbf{v}}', I)) &= \alpha_{\underline{\mathbf{0}}}(\underline{\mathbf{v}} + \underline{\mathbf{v}}', I)\\ &= \underline{\mathbf{v}} + \underline{\mathbf{v}}'\\ &= \alpha_{\underline{\mathbf{0}}}((\underline{\mathbf{v}}, I))\alpha_{\underline{\mathbf{0}}}((\underline{\mathbf{v}}', I))\end{aligned}$$

(note + is the group law of $\mathbb{R}^2$). Further more $\alpha_{\underline{\mathbf{0}}}$ is surjective since $\forall \underline{\mathbf{v}} \in \mathbb{R}^2$,

$$\alpha_{\underline{\mathbf{0}}}((\underline{\mathbf{v}}, I)) = \underline{\mathbf{v}}$$

hence $\alpha_{\underline{\mathbf{0}}}$ is an isomorphism since $|T| = |\mathbb{R}^2|$ and $\alpha_{\underline{\mathbf{0}}}$ is injective. There for L is a subgroup of $\mathbb{R}^2$ since $\alpha_{\underline{\mathbf{0}}}$ sends H to L and H is a subgroup of T. The points spanned by $m\underline{\mathbf{a}} + n\underline{\mathbf{b}}\, m, n \in \mathbb{Z}$ are therefore in L. Using the points of this lattice we can divide the plane into parallelograms as in figure 6. Assume there exists a point $\underline{\mathbf{x}} \in L$ which does not lie on the lattice, then the point is in a parallelogram or on the edge of two adjacent parallelograms, in the latter case pick either of the two parallelograms. Then choose the corner, $\underline{\mathbf{c}}$, of this parallelogram which is closest to $\underline{\mathbf{x}}$, the vector $\underline{\mathbf{x}} - \underline{\mathbf{c}}$ is not $\underline{\mathbf{0}}$ or $\underline{\mathbf{a}}$ or $\underline{\mathbf{b}}$ and is in L since L is a group. Furthermore, $\|\underline{\mathbf{x}} - \underline{\mathbf{c}}\| < \|b\|$. If $\|\underline{\mathbf{x}} - \underline{\mathbf{c}}\| < \|a\|$ then this is a contradiction since $\underline{\mathbf{a}}$ is supposed to be a minimum in L. If $\underline{\mathbf{a}} \leq \|\underline{\mathbf{x}} - \underline{\mathbf{c}}\| < \|b\|$ then $\underline{\mathbf{x}} = \underline{\mathbf{c}}$ is linearly independent of $\underline{\mathbf{a}}$ which is a contradiction since $\underline{\mathbf{b}}$ is the smallest vector linearly independent from $\underline{\mathbf{a}}$. Therefore no such $\underline{\mathbf{x}}$ exists thus, $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ span L. □

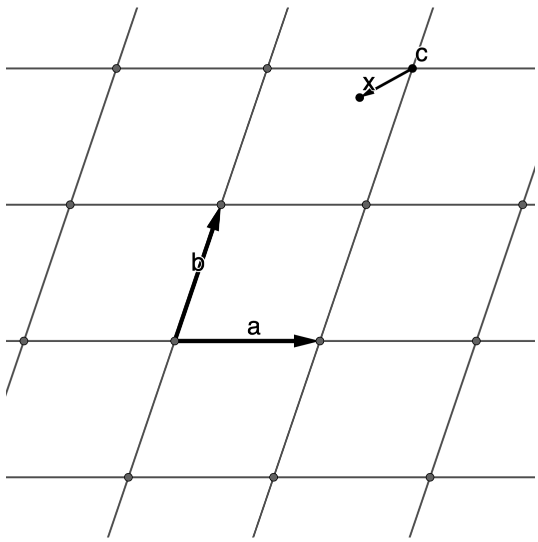



Figure 6

We now can classify each type of lattice based on their shape which depends on $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$. Firstly replace $\underline{\mathbf{b}}$ with $-\underline{\mathbf{b}}$ if necessary to ensure

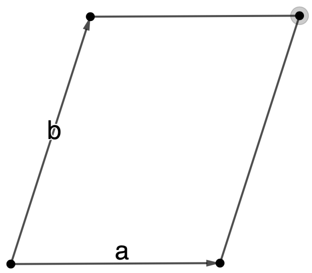
$$\|\underline{\mathbf{a}} - \underline{\mathbf{b}}\| \leq \|\underline{\mathbf{a}} + \underline{\mathbf{b}}\|$$

We now define each type of lattice,

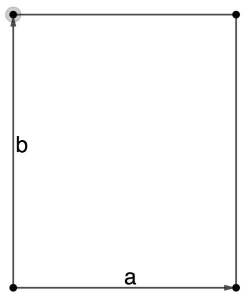
**Definition 3.4.** The 5 types of lattices are:
(a) **oblique** $\|\underline{\mathbf{a}}\| < \|\underline{\mathbf{b}}\| < \|\underline{\mathbf{a}} - \underline{\mathbf{b}}\| < \|\underline{\mathbf{a}} + \underline{\mathbf{b}}\|$
(b) **rectangular** $\|\underline{\mathbf{a}}\| < \|\underline{\mathbf{b}}\| < \|\underline{\mathbf{a}} - \underline{\mathbf{b}}\| = \|\underline{\mathbf{a}} + \underline{\mathbf{b}}\|$

(c) **centered rectangular** $\|\underline{\mathbf{a}}\| < \|\underline{\mathbf{b}}\| = \|\underline{\mathbf{a}} - \underline{\mathbf{b}}\| < \|\underline{\mathbf{a}} + \underline{\mathbf{b}}\|$ (or $\|\underline{\mathbf{a}}\| = \|\underline{\mathbf{b}}\| < \|\underline{\mathbf{a}} - \underline{\mathbf{b}}\| < \|\underline{\mathbf{a}} + \underline{\mathbf{b}}\|$)
(d) **square** $\|\underline{\mathbf{a}}\| = \|\underline{\mathbf{b}}\| < \|\underline{\mathbf{a}} - \underline{\mathbf{b}}\| = \|\underline{\mathbf{a}} + \underline{\mathbf{b}}\|$
(e) **hexagonal** $\|\underline{\mathbf{a}}\| = \|\underline{\mathbf{b}}\| = \|\underline{\mathbf{a}} - \underline{\mathbf{b}}\| < \|\underline{\mathbf{a}} + \underline{\mathbf{b}}\|$

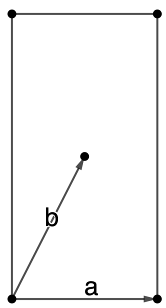
This covers every possibility. In the centered rectangular case there are two possibilities for the second the parallelogram become rhombus. The diagonals of the rhombus bisect at right angles so we have a c.rectangular structure where the rectangles are the vectors $\underline{\mathbf{a}} - \underline{\mathbf{b}}$ and $\underline{\mathbf{a}} + \underline{\mathbf{b}}$. These structures will become clear in the following diagrams.



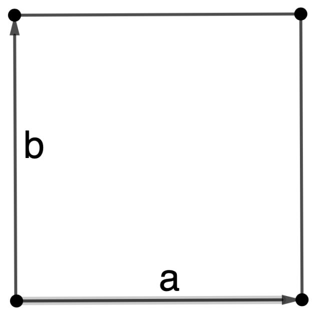

(a) Oblique



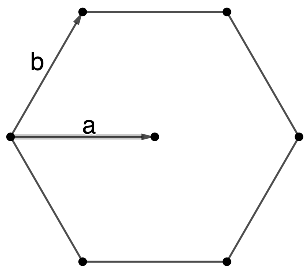

(b) Rectangular




(c) Centered rectangular




(d) Square




(e) Hexagonal

Figure 7

So far Theorem 3.3 tells us that for the translations of a wallpaper group we can restrict our view from the whole plane to just points on the lattice, however we still don't know why this is useful since the wallpaper group has more than just translations. The following theorem is extremely important as it tells us that we can completely restrict our view to the lattice only.

**Theorem 3.5.** *For a wallpaper group G with point group J and lattice L, J acts on L. That is elements of J preserve the lattice.*

*Proof.* The point group J is a subgroup of $O_2$ since it is the image of the homomorphism 3.1, hence elements of J act on the plane as orthogonal transformations. We must show if $M \in J$ then for all $\underline{\mathbf{x}} \in L$, $f_M\underline{\mathbf{x}} \in L$. Suppose $g \in G$ is such that $g = (\underline{\mathbf{v}}, M)$ so $\pi(g) = M \in J$ and let $\tau = (\underline{\mathbf{x}}, I) \in G$. Since $\tau \in H$ and $H$ is the kernel of $\pi$ hence a normal subgroup of $G$, $g\tau g^{-1} \in H$. Then,

$$\begin{aligned} g\tau g^{-1} &= (\underline{\mathbf{v}}, M)(\underline{\mathbf{x}}, I)(-f_M^{-1}(\underline{\mathbf{v}}), M^{-1}) \\ &= (\underline{\mathbf{v}}, M)(\underline{\mathbf{x}} - f_M^{-1}(\underline{\mathbf{v}}), M^{-1}) \\ &= (\underline{\mathbf{v}} + f_M(\underline{\mathbf{x}} - f_M^{-1}(\underline{\mathbf{v}})), MM^{-1}) \\ &= (f_M(\underline{\mathbf{x}}), I) \end{aligned}$$

Therefore $f_M(\underline{\mathbf{x}})$ is on the lattice. □

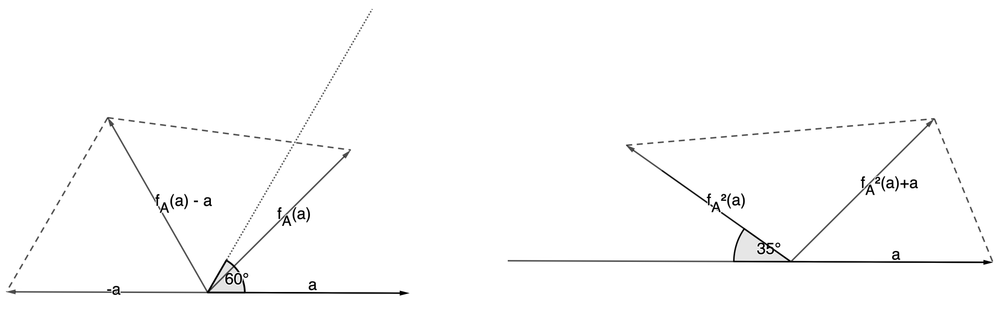
Notice we cannot say the wallpaper group G acts on the lattice L since the elements of G do not necessarily preserve L. For example consider the wallpaper group with a translation $\tau(x, y) = (x, y+1)$ and glide reflection $g(x, y) = (-x, y+1)$ then the point group J consists of the identity matrix and reflection matrix both of which preserve L by 3.5, however the glide reflection will not send points of the lattice onto points of the lattice. Now we can really start restricting possible elements of the point group J known as crystallographic restriction Coxeter (1989).

**Theorem 3.6.** *The order of a rotation in a wallpaper group can only be 2, 3, 4 or 6*

*Proof.* Every rotation of a wallpaper group has finite order by definition 3.1 since the point group is finite. If we have a rotation of order $q \in \mathbb{Z}^+$ then a suitable rotation matrix is,

$$A = \begin{bmatrix} \cos\frac{2\pi}{q} & -\sin\frac{2\pi}{q} \\ \sin\frac{2\pi}{q} & \cos\frac{2\pi}{q} \end{bmatrix}$$

and is in the point group $J$. We know from 3.5 that $J$ acts on $L$ so $f_A(\underline{\mathbf{a}}) \in L$ where $\underline{\mathbf{a}}$ is the vector of shortest length in the lattice. Suppose $q$ is greater than 6, then $\frac{2\pi}{q}$ is less than $\frac{\pi}{3} = 60°$ and $f_A(\underline{\mathbf{a}}) - \underline{\mathbf{a}}$ is a vector in $L$ which is shorter than $\underline{\mathbf{a}}$ contradicting that $\underline{\mathbf{a}}$ is the shortest vector in L see figure 8a. Hence $q \leq 6$. Suppose $q = 5$ then our angle is $\frac{2\pi}{5} = 72°$, then $f_A^2(\underline{\mathbf{a}}) + \underline{\mathbf{a}}$ is in $L$ and is shorter than $\underline{\mathbf{a}}$ see figure 8b. Hence the only values for $q$ are 2, 3, 4 or 6.




(a) (b)

Figure 8

∎

We can restate this theorem,

**Corollary 3.6.1.** *The point group $J$ of a wallpaper group $G$ is generated by a rotation through an one of the angles $0$, $\pi$, $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{3}$ and possibly a reflection.*

*Proof.* As in 3.6 the the rotations of the point group must have order 2, 3, 4 or 6 hence angles $0, \pi, \frac{2\pi}{3}, \frac{\pi}{2}$ or $\frac{\pi}{3}$. We cannot have two or more rotation matrices of different order in the point group since composing them would create a rotation of a not allowed order, and if we had a rotation of order 2 and of order 4, we can simplify this by using the composition to give a rotation of order 6 which generates the the point group. We also may have one reflection. Suppose there were two reflections, If they were parallel they would appear the same in the point group. If they are not parallel then we could generate one from the other by composing with a rotation, If there is no such composition then by 2.5.3 we could generate a rotation which was not already in the point group by composing the two mirrors. Therefore the point group only needs to be generated by one mirror ∎

Note this does not necessarily mean the point group can only have one reflection just that all the reflections are compositions of another reflection and a rotation. We now have enough theory to start to enumerate the 17 wallpaper groups. We just need some results to help distinguish if two given wallpaper groups are isomorphic so we do not double count the groups we find.

**Theorem 3.7.** *An isomorphism between wallpaper groups sends translations to translations, rotations to rotations, reflections to reflections and glide reflections to glide reflections.*

*Proof.* Let $\phi : G \to G'$ be an isomorphism between two wallpaper groups and let $\tau \in G$ be a translation. Translations and glides have infinite order, rotations and reflections are of finite order thus the proof splits to two cases. Firstly $\phi(\tau)$ must be either a translation or glide. Assume $\phi(\tau)$ is a glide reflection and choose a translation $\tau' \in G'$ whose direction is not parallel to the line of glide then,

$$\begin{aligned}\phi(\tau)\tau' &= (2\underline{(a)} + \underline{(b)}, B)(\underline{(v)}', I) && \textit{forms from section 2.4}\\ &= (2\underline{\mathbf{a}} + \underline{\mathbf{b}} + f_B(\underline{\mathbf{v}}'), BI)\\ &= (2\underline{\mathbf{a}} + \underline{\mathbf{b}} + f_B(\underline{\mathbf{v}}'), B)\\ \textit{and}\\ \tau'\phi(\tau) &= (\underline{(v)}', I)(2\underline{(a)} + \underline{(b)}, B) && \textit{forms from section 2.4}\\ &= (\underline{\mathbf{v}}' + f_I(2\underline{\mathbf{a}} + \underline{\mathbf{b}}), IB)\\ &= (\underline{\mathbf{v}}' + 2\underline{\mathbf{a}} + \underline{\mathbf{b}}, B)\end{aligned}$$

hence $\tau'$ commutes with $\phi(\tau)$ if and only if $f_B(\underline{\mathbf{v}}') = \underline{\mathbf{v}}'$ that is if $\tau$ is parallel to the line of glide, therefore $\tau'$ does not commute with $\phi(\tau)$. If $\phi(g) = \tau'$, then $g$ is a translation or glide. So $g^2$ is a translation since the composition of two glides is a translation. Hence $g^2$ commutes with $\tau$. This is a contradiction since we know $\phi$ is a homomorphism and $\phi(g^2) = \tau'^2$ does not commute with $\phi(\tau)$ because $\tau'^2$ is in the same direction as $\tau'$. Therefore translations are sent to translations and glides to glides. For the case of finite order elements note that reflections have order 2 so can only correspond to reflections or rotations by angle $\pi$. Let $g \in G$ be a reflection and assume $\phi(g)$ is a rotation by angle $\pi$. Choose a translation $\tau \in G$ which is parallel to the line of reflection of g then $\tau g$ is a glide. But then $\phi(\tau g) = \phi(\tau)\phi(g)$ is the product of a translation and a rotation which is a rotation. This is a contradiction since we know glides cannot map to rotations. Hence reflections map to reflections and consequently rotations must map to rotations. □

Before the final result we recall the definition cosets and quotient groups.

**Definition 3.8.** For a group G with subgroup H. For a fixed $a \in G$ a **right H-coset of G** is defined:

$$Ha = \{ha \mid h \in H\}$$

**Theorem 3.9.** *For a group G with subgroup H. The set of all right H-cosets written $[G : H]$ form a partition of the group G. Furthermore any two cosets, $Ha, Hb \in [G : H]$ are disjoint or equal.*

**Theorem 3.10.** *For a group G with normal subgroup H. The set of all cosets, $[G : H]$ with the binary operation defined:*

$$\begin{gathered}[G : H] \times [G : H] \to [G : H]\\ Ha \circ Hb \mapsto Hab\end{gathered}$$

*forms a group.*

The proof for both of these theorems is part of a standard group theory course and so will be omitted, they can be found in Armstrong (1988).
Recall back to the start of the chapter and notice the homomorphism 3.1 is surjective when restricted to $G$, hence the quotient group $G/H$ is isomorphic to the point group $J$.

**Corollary 3.10.1.** *If two wallpaper groups $G$, $G_1$ are isomorphic then their point groups $J$, $J_1$ are isomorphic.*

*Proof.* Let G and G′ be wallpaper groups with translation subgroups H and H′ and point groups J and J′ respectively. Suppose $\phi : G \to G'$ is an isomorphism, then $\phi(H) = H'$ by theorem 3.7. Therefore there is an isomorphism between G/H and G′/H′ thus,

$$J \cong G/H \cong G'/H' \cong J'$$

□

### 3.3 Examples

Before looking at how to enumerate the wallpaper groups we will see some examples.

**Example 3.3.1.** Is the subgroup of $E_2$ generated by the glide reflections $g(x, y) = (-x, y+1)$ and $g(x, y) = (-x+2, y+1)$ a wallpaper group? Recall the definition 3.1 we need exactly two independent translations and the point group to be finite. In this case we show there are at least 3 independent translations. We know the composition of a glide reflection with its self makes a translation so,

$$\begin{aligned} g \circ g(x, y) &= g(g(x, y)) \\ &= (-(-x), (y+1)+1) \\ &= (x, y+2) \\ and \\ h \circ h(x, y) &= h(h(x, y)) \\ &= (-(-x+2)+2, (y+1)+1) \\ &= (x, y+2) \end{aligned}$$

so we get only one independent translation. However, in this case this shows these two glide reflections are parallel so we can compose them with each other to get another translation,

$$\begin{aligned} g \circ h(x, y) &= g(h(x, y)) \\ &= (-(-x)+2, (y+1)+1) \\ &= (x+2, y+2) \\ and \\ h \circ g(x, y) &= h(g(x, y)) \\ &= (-(-x+2), (y+1)+1) \\ &= (x-2, y+2) \end{aligned}$$

this gives two more independent translations hence there are at least three independent translations so we cannot have a wallpaper group.

**Example 3.3.2.** Is the subgroup of $E_2$ generated by the translation $\tau = (x, y+1)$, the reflection in the x-axis and the reflection in the line $x = 1$ a wallpaper group? In this case we already have one translation but need to find another. From example 2.5.3 we know the composition of two parallel mirrors makes a translation so since the reflection in the x-axis is $g(x, y) = (-x, y)$ and the translation in the line $x = 1$ is $h(x, y) = (-x+1, y)$,

$$\begin{aligned} g \circ h(x, y) &= g(h(x, y)) \\ &= (-(-x+1), y) \\ &= (x-1, y) \\ and \\ h \circ g(x, y) &= h(g(x, y)) \\ &= (-(-x)+1, y) \\ &= (x+1, y) \end{aligned}$$

This adds just one more independent translation since $(x+1, y)$ and $(x-1, y)$ are linearly dependent. There are no other ways to get another translation so all we need to check is the point group is finite. In this case the point group is given by,

$$J = \{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\}$$

which is finite so we do indeed have a wallpaper group.

**Example 3.3.3.** Let $\tau$ be a translation by $\underline{\mathbf{v}}$ and $g$ be a reflection in the line m. Prove $g\tau$ is a reflection if $\underline{\mathbf{v}}$ is perpendicular to m and a glide reflection otherwise. Using notation from section 2.4 we can write $\tau = (\underline{(v)}, I)$ and $g = (2\underline{\mathbf{a}}, B)$. If $\tau$ is perpendicular to m then $f_B(\underline{\mathbf{v}}) = -\underline{\mathbf{v}}$. So when $\tau$ is perpendicular to m,

$$\begin{aligned}
g\tau &= (2\underline{\mathbf{a}}, B)(\underline{\mathbf{v}}, I)\\
&= (2\underline{\mathbf{a}} + f_B(\underline{\mathbf{v}}), BI)\\
&= (2\underline{\mathbf{a}} - \underline{\mathbf{v}}, B) && \textit{since } \tau \textit{ is perpendicular to m}\\
&= (2(\underline{\mathbf{a}} - \frac{\underline{\mathbf{v}}}{2}), B)
\end{aligned}$$

this is a reflection in the line parallel to m, $\underline{(a)} - \frac{\underline{\mathbf{v}}}{2}$ from the origin. If $\tau$ is not perpendicular to m then following ideas from 2.7 we can set $\underline{\mathbf{w}} = 2\underline{\mathbf{a}} + f_B(\underline{\mathbf{v}}) - f_B(2\underline{\mathbf{a}} + f_B(\underline{\mathbf{v}}))$ then $f_B(\underline{\mathbf{w}}) = -\underline{\mathbf{w}}$. we then resolve $2\underline{\mathbf{a}} + f_B(\underline{\mathbf{v}})$ along $\underline{\mathbf{w}}$ and we have a glide reflection.

# 4 Enumeration of wallpaper patterns

## 4.1 General method

This chapter takes ideas from Armstrong (1988), descriptions of other wallpaper groups can be found there. We start to enumerate the wallpaper groups by looking at each type of lattice in turn and finding the orthogonal transformations which preserve that lattice, which will form a group. Therefore we know the point group of a wallpaper group will be a subgroup of these transformations by 3.5. We can then deduce the different wallpaper groups which could arise from the point groups. Lastly we need to check once we have found all the possible wallpaper groups this way then no two are isomorphic. There are 17 different wallpaper groups and and they are named following a notation; letters p, c, m, g and integers 1, 2, 3, 4, 6. The numbers indicate the order of possible rotations of the pattern (1 indicates no rotation), p refers to a lattice with no central lattice point and c refers to a lattice with a central point such as the centered rectangular, m indicates a possible mirror and g a glide. Recall G is a wallpaper group with point group J, translation subgroup H and lattice L. For convenience we can take the vector $\underline{\mathbf{a}}$ which defines the lattice to lie in the same direction as the positive x-axis, then we can take $\underline{\mathbf{b}}$ to lie in the first quadrant. Therefore when trying to enumerate the wallpaper groups we can restrict our view to just the fundamental parallelogram of the lattice. Or general approach for enumerating each wallpaper group goes as follows:

- Choose a lattice.
- Find the point group which preserves the lattice.
- Choose a subgroup of the point group.
- If there is a reflection element in the point group, realise this as a standard mirror or glide.
- Find all of the elements of the wallpaper group with this point subgroup.

Practically it will be easier to look at an example.

## 4.2 Rectangular lattice

Lets first look at the rectangular lattice. we need to find which orthogonal transformations preserve the lattice. So given a general point $\underline{\mathbf{x}} = m\underline{\mathbf{a}} + n\underline{\mathbf{b}} \in L$, where $m, n \in \mathbb{Z}$, we need to find which orthogonal matrices send $\underline{\mathbf{x}}$ to a point on $L$. Since any orthogonal transformation does not scale vectors in the lattice we need only look at transformations which send $\underline{\mathbf{a}}$ to $\underline{\mathbf{a}}$ or $-\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ to $\underline{\mathbf{b}}$ or $-\underline{\mathbf{b}}$ (If $\underline{\mathbf{a}}$ or $\underline{\mathbf{b}}$ are mapped to any other vector then they can not be on the lattice.). Its clear that the identity will preserve the lattice since it sends $\underline{\mathbf{x}}$ to $\underline{\mathbf{x}}$ which is on the lattice. Similarly the matrix which represents a rotation by $\pi$ is $-I$ which sends $\underline{\mathbf{x}}$ to $-\underline{\mathbf{x}}$ which is always on $L$ (These are the only two orthogonal transformations which preserve every type of lattice). For a rectangular lattice we also have the reflections $B_0$ and $B_\pi$ where $B_\phi$ is the matrix in 2.5. This is clear since $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are perpendicular for the rectangular lattice and we can take $\underline{\mathbf{a}}$ to lie in the direction of the positive x-axis. So if there is a reflection in the x-axis ($B_0$) then $\underline{\mathbf{a}}$ remains the same and $\underline{\mathbf{b}}$ is sent to $-\underline{\mathbf{b}}$ so the general point $\underline{\mathbf{x}} = m\underline{\mathbf{a}} + n\underline{\mathbf{b}}$ goes to $m\underline{\mathbf{a}} - n\underline{\mathbf{b}}$ which is on $L$ since $-n \in \mathbb{Z}$. A similar argument is made for $B_\pi$ but the roles of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are reversed. Therefore the set of orthogonal transformations which preserve a rectangular lattice is $\{I, -I, B_0, B_\pi\}$.

**pm**

Now we need to look at subgroups of this group. We will take $\{I, B_0\}$. There are going to be translations and reflections. The translations will clearly be of the form $(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, I)$. Assume $B_0$ is realised as a mirror and not a glide in G. We can choose the origin to lie on a mirror hence $(\underline{\mathbf{0}}, B_0) \in G$. We will now use a technique which makes use of the H-cosets where H is the translation subgroup of G. Recall the set of all H-cosets form a disjoint union of G and so if we can find all the cosets we find the whole of G. Also recall there is an isomorphism between the point group and the quotient group $G/H$ (The quotient group is $G/H$ is the set of all H-cosets). So we conclude we need to find two different H-cosets then we have the whole of G since the point group has 2 elements. We have the following two cosets,

$$H(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, I) = H$$
$$H(\underline{\mathbf{0}}, B_0) \neq H$$

These are disjoint since we have no reflections in $H$. We conclude the elements of this wallpaper group are of the form:

$$(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, I) \quad \textit{translations}$$
$$(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, B_0) \quad \textit{reflections}$$

recall back to 2.4 to see that the rotations in this case are the rotations by $\pi$ about the points $\frac{1}{2}m\underline{\mathbf{a}} + \frac{1}{2}n\underline{\mathbf{b}}$ .If we take the point group as the subgroup $\{I, B_\pi\}$ we will get the same description of elements in the group but with a reflection in the vertical, this leads to isomorphic groups.

**p2mm**

Let the point group be the subgroup $\{I, -I, B_0, B_\pi\}$ and let both reflections be realised as mirrors not glides. We need to find four disjoint H-cosets. We can take the origin to be at the intersection of a horizontal mirror and a vertical mirror. So both the elements $(\underline{\mathbf{0}}, B_0), (\underline{\mathbf{0}}, B_\pi) \in G$. Since $G$ is a group we also have,

$$(\underline{\mathbf{0}}, B_0)(\underline{\mathbf{0}}, B_\pi) = (\underline{\mathbf{0}}, -I) \in G$$

therefore our four H-cosets are,

$$H$$
$$H(\underline{\mathbf{0}}, B_0)$$
$$H(\underline{\mathbf{0}}, B_\pi)$$
$$H(\underline{\mathbf{0}}, -I)$$

Which are clearly disjoint by looking at the matrix component. Therefore the elements are of the form:

$$(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, I) \quad \textit{translations}$$
$$(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, -I) \quad \textit{rotations by } \pi$$
$$(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, B_0) \quad \textit{reflections in horizontal}$$
$$(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, B_\pi) \quad \textit{reflections in vertical}$$

The other wallpaper group for the a rectangular lattice are **pg, p2mg, p2gg.** Descriptions of these can be found in Armstrong (1988).

### 4.3 Hexagonal lattice

The orthogonal transformations which preserve this lattice are the rotations generated by $A_{\frac{\pi}{3}}$ and a reflection $B_0$, since in this case $\underline{\mathbf{a}}$ can be sent to $\pm\underline{\mathbf{a}}, \pm\underline{\mathbf{b}}$ or $\pm(\underline{\mathbf{a}} - \underline{\mathbf{b}})$.
**p3**
Let the point group be the subgroup, $J = \{I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}\}$. We can take the origin to be a centre of rotation so, the points $(\underline{\mathbf{0}}, A_{\frac{2\pi}{3}}), (\underline{\mathbf{0}}, A_{\frac{4\pi}{3}})$ are in $G$. Since we are only looking for 3 H-cosets we know the elements of G are:

$$\begin{aligned}
&(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, I) \quad \textit{translations} \\
&(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, A_{\frac{2\pi}{3}}) \quad \textit{rotations by } \frac{2\pi}{3} \\
&(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}, A_{\frac{4\pi}{3}}) \quad \textit{rotations by } \frac{4\pi}{3}
\end{aligned}$$

We must not naively think of these elements as rotations about points on the lattice, we must recall back to subsection 2.4 to see an element of the form $(\underline{\mathbf{c}} - f_A(\underline{\mathbf{c}}), A)$ is a rotation about the point $\underline{\mathbf{c}}$. So in this case we need to solve for $\underline{\mathbf{c}}$,

$$\begin{aligned}
&m\underline{\mathbf{a}} + n\underline{\mathbf{b}} = \underline{\mathbf{c}} - f_A(\underline{\mathbf{c}}) \\
\implies &\underline{\mathbf{c}} = f_{I-A}^{-1}(m\underline{\mathbf{a}} + n\underline{\mathbf{b}}) = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & -(1+\frac{\sqrt{3}}{2}) \\ -1+\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} m\underline{\mathbf{a}} + n\underline{\mathbf{b}}
\end{aligned}$$

### 4.4 17 non-isomorphic wallpaper groups

The 17 wallpaper groups are called: **p1, p2, pm, pg, p2mm, p2mg, p2gg, cm, c2mm, p4, p4mm, p4gm, p3, p3m1, p31m, p6, p6mm.** We have described the elements of some of these groups above and more descriptions can be found in Armstrong (1988). We still need to. prove that no two of these groups are isomorphic. Firstly we know by 3.10.1 that if two of these groups are isomorphic then their point groups are isomorphic, hence we only need to compare the groups with isomorphic point groups. We shall take an example from Armstrong (1988).

**Theorem 4.1.** *No two of **p2, pm, pg, cm** are isomorphic.*

*Proof.* Only **p2** contains rotations hence by theorem 3.7 cannot be isomorphic to the others. From the remaining groups only **pg** has no reflections and just glide reflections so is not isomorphic to **pm or cm**. Finally, if we take a glide in **pm** it can be written as a reflection followed by a translation and both the reflection and the translation are elements in the group. However, **cm** contains glides which cannot be written as a reflection followed by a translation using only elements in the group. □

Proofs the other wallpaper groups are non-isomorphic are found in Armstrong (1988).

### 4.5 Conclusion

We can now come full circle and describe what kind of symmetry is exhibited in Escher's works. First we will look at figure 1. This is not the best example for two reasons: it is very artistic which distracts from the underlying repeating pattern we are trying to decode and, also its not a very 'flashy' wallpaper group (more of a subjective preference). Just taking the pattern for what it is we get the wallpaper group **p1** generated by just two independent translations as seen in figure 9 since if we look at one of the white birds it cannot rotate around any point to map onto another white bird and similarly it cannot reflect onto another white bird as it would be facing the wrong way. However if we ignore that the white birds and black birds are different then we can notice the white birds can be glide reflected down the blue vertical line onto a black bird. If we include this glide reflection we get the the wallpaper group **pg**.

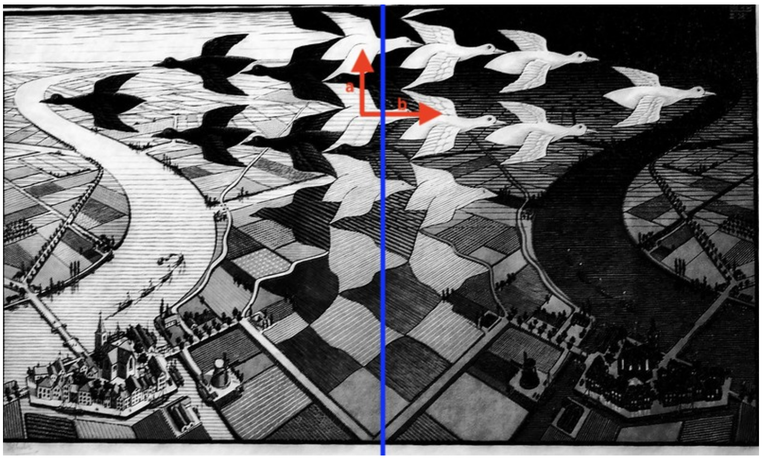



Figure 9: Escher (1938)

We will look at one more example which is more illustrative.

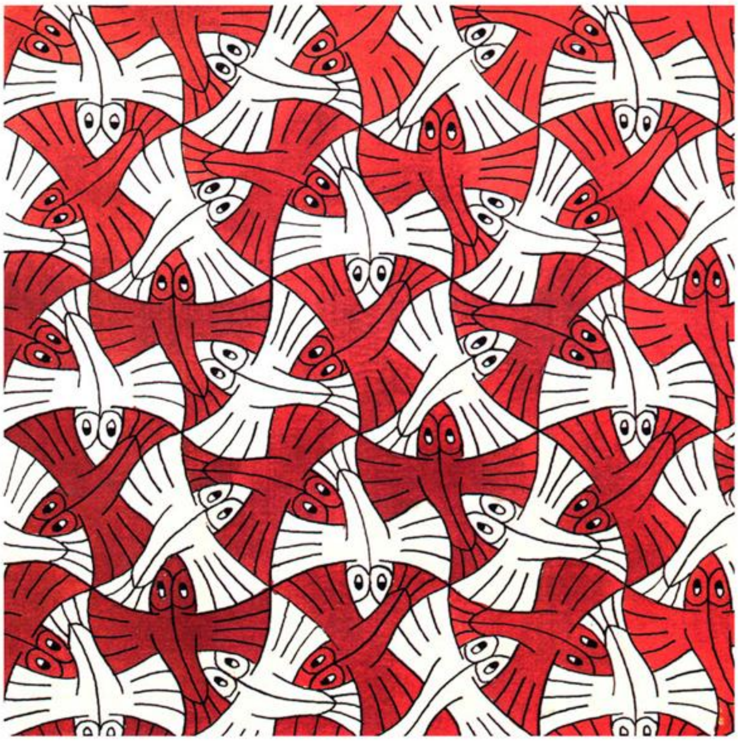


Figure 10: Escher (1948)

In figure 10 we see we can rotate about the point where the wings of 6 fish meet and the image will remain the same; if we do not ignore the colours of the fish then this rotation of order 3, by $\frac{2\pi}{3}$. Also notice there are no reflections or glide reflections. Therefore, we must have the group **p3**. If we ignored the colours of the fish we would have a rotation of order 6, by angle $\frac{\pi}{3}$. There are still no reflections so we have the group **p6**.

# 5 Wallpaper patterns in different geometries

## 5.1 Introduction

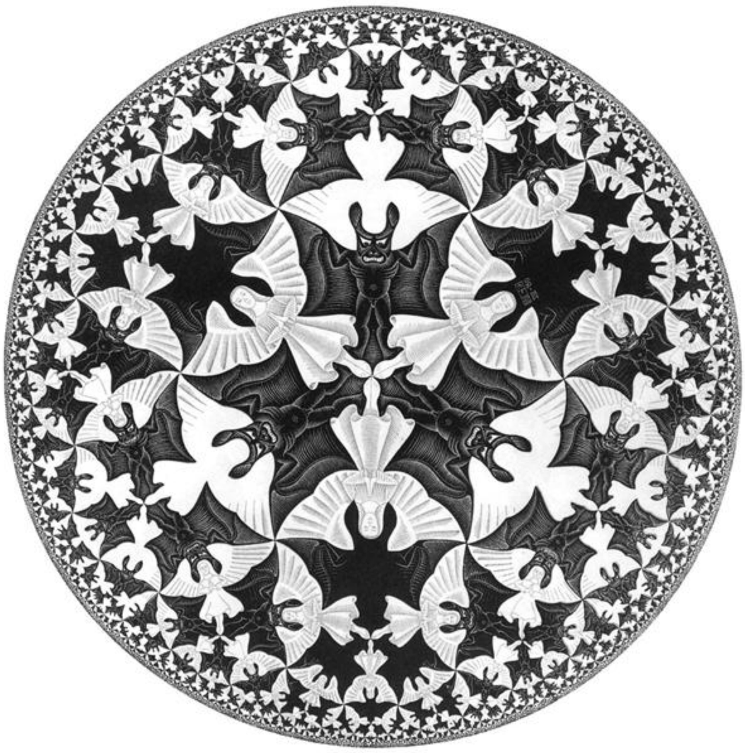


Figure 11: Escher (1960)

When first looking at Escher's works for this project I came across figure 11. I was curious since it had a similar kind of regular symmetry but was clearly not part of any wallpaper groups detailed above. I when reading into this topic I also had some unanswered questions. One of these questions was regarding the definition of isometries, definition 2.1. What if we could change the metric for measuring distance in this definition? It turns out that question was slightly naive and what I was really trying to ask was, do we exhibit similar symmetry patterns in different geometries? As it happens this question and figure 11 led me down the same path, the symmetry groups of hyperbolic geometry.

## 5.2 The Poincaré disc

Before we start to think about patterns in hyperbolic geometry we need to define a hyperbolic space to work in. Although not the only model for 2 dimensional hyperbolic space, the Poincaré disc is the best in this case since figure 11 is constructed in the Poincaré disc.

Other models include the Poincaré half plane model, Klein model and the Lorentz model. Hyperbolic geometry takes the 5 postulates from Euclidean geometry but replaces the parallel postulate with a different definition loosely stating there exist at least two lines through a point that are both parallel to a common line. We will interchange between $(x, y)$ notation and complex notation. Hyperbolic geometry in the Poincaré disc consists only of points in the unit disc $\mathfrak{D}$,

$$\mathfrak{D} = \{z \mid |z| < 1\} = \{(x, y) \mid x^2 + y^2 < 1\}$$
$$\partial\mathfrak{D} = \{z \mid |z| = 1\} = \{(x, y) \mid x^2 + y^2 = 1\} \qquad z \in \mathbb{C}, x, y \in \mathbb{R}$$

The points on the boundary, $\partial\mathfrak{D}$, do not belong to the geometry. We can define a line in this geometry. Brannan (2011)

**Definition 5.1.** A line of hyperbolic geometry is part of a generalised (Euclidean) circle which meets $\partial\mathfrak{D}$ at right angles and lies in $\mathfrak{D}$.

Examples of hyperbolic lines are given in figure 12. We can ignore the lines outside of $\partial\mathfrak{D}$, they are there to help demonstrate that the hyperbolic lines are parts of a Euclidean circle.

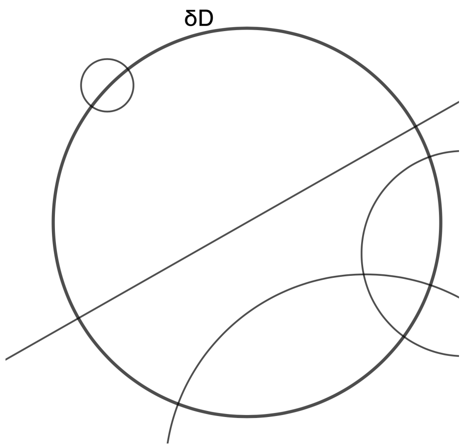



Figure 12: Examples of lines on a Poincare disc

We want to follow similar motivation as we did for the Euclidean plane so we want to find functions which preserve some 'distance' in the hyperbolic model. Before we can find these functions we need to know what is the 'distance' which needs to be preserved. We can find how to measure the distance between two points using our definition of hyperbolic lines. One of the other postulates for hyperbolic geometry is that for any two points there is a unique line connecting these points. With this information we can motivate the definition of hyperbolic distance. Hvidsten (2016).

**Definition 5.2.** The hyperbolic distance between two points A = $z_0$ and B = $z_1$ on the Poincaré disc, called **Poincaré distance**, is given by,

$$d(A, B) = |\ln(\frac{(AF)(BE)}{(AE)(BF)})|$$
$$= |\ln(\frac{|z_0 - w_1||z_1 - w_0|}{|z_0 - w_0||z_1 - w_1|})|$$

Where the E = $w_0$ and F = $w_1$ are the points where the unique hyperbolic line connecting A and B meet $\partial\mathfrak{D}$, and AF is the Euclidean distance between A and F, similarly for BE, AE and BF.

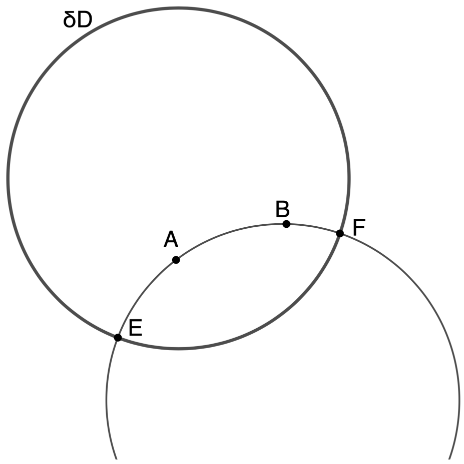



Figure 13

So the aim is to find the set of all maps from the unit disc to itself which preserve the Poincaré distance, then hopefully this will form a group and we can proceed as in the Euclidean case.

### 5.3 Möbius transformations and cross ratio

From here on out we will stick with complex notation as it is more convenient. It turns out the functions we are looking for are Möbius transformations defined below. Hvidsten (2016)

**Definition 5.3.** A **Möbius transformation**, f, is defined:

$$f : \mathbb{C} \to \mathbb{C}$$
$$f(z) \mapsto \frac{az+b}{cz+d}$$

With $a, b, c, d \in \mathbb{C}$ and $ab - cd \neq 0$. The set of all Möbius transformations is called $Mob(\mathbb{C})$

Why do we consider functions such as these? Möbius transformations are the set of all one-to-one conformal maps. A conformal map loosely is a map which preserves the Euclidean notion of angle but not necessarily the Euclidean notion of length. They have many other applications in geometry which can be seen in Kisil (2012). We still need to prove that these transformations preserve the Poincaré distance 5.2 but to do that we first develop some theory about Möbius transformations.

**Theorem 5.4.** *The set of all Möbius transformations defined in definition 5.3 form a group with the group law of composition of functions.*

*Proof.* We need to satisfy the group axioms: identity, closure, inverses, associativity. Associative follows since function composition is associative.
Identity:

The identity transformation is the usual identity map $id$ since for any Möbius transformation $g \in Mob(\mathbb{C})$,

$$id \circ g = g \circ id = g$$

Inverses:
For a Möbius transformation $f(z) = \frac{az+b}{cz+d}$ let $g(z) = \frac{-dz+b}{cz-a} \in Mob(\mathbb{C})$ then,

$$\begin{aligned} f \circ g &= \frac{a(\frac{-dz+b}{cz-a}) + b}{c(\frac{-dz+b}{cz-a}) + d} \\ &= \frac{-cdz + ab + b(cz-a)}{-cdz + cb + d(cz-a)} \\ &= z \\ &= id(z) \\ and & \\ g \circ f &= \frac{-d\frac{az+b}{cz+d} + b}{c\frac{az+b}{cz+d} - a} \\ &= \frac{-daz - db + b(cz+d)}{caz + cb - a(cz+d)} \\ &= z \\ &= id(z) \end{aligned}$$

hence g is the inverse of f.
Closure:
For any two $f, g \in Mob(\mathbb{C})$, $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{a'z+b'}{c'z+d'}$,

$$\begin{aligned} f \circ g &= \frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} \\ &= \frac{(aa'bc')z + (b'bad')}{(ca'dc')z + (cb'dd')} \end{aligned}$$

Which is a Möbius transformation, hence the Möbius transformations form a group. □

Just as the Euclidean isometries are made from translations, rotations and reflections the Möbius transformations are made from translations, dilations and inversions.

**Definition 5.5.** A **dilation** is a complex map,

$$\begin{aligned} d &: \mathbb{C} \to \mathbb{C} \\ d(z) &\mapsto az \end{aligned}$$

for $a \in \mathbb{C}$.
A **inversion** is the complex map,

$$\begin{aligned} in &: \mathbb{C} \to \mathbb{C} \\ in(z) &\mapsto \frac{1}{z} \end{aligned}$$

**Theorem 5.6** (Hvidsten (2016)). *A Möbius transformation is always the composition of translations, inversions and dilations.*

*Proof.* For a Möbius transformation $f(z) = \frac{az+b}{cz+d}$, if $c = 0$ then,

$$f(z) = \frac{a}{d}z + \frac{b}{d}$$

which is a dilation by $\frac{a}{d}$ followed by a translation by $\frac{b}{d}$.
If $c \neq 0$ then,

$$f(z) = \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}}$$

which is the composition of a translation by $\frac{d}{c}$ followed by an inversion then a dilation by $-\frac{ad-bc}{c^2}$ and finally a translation by $\frac{a}{c}$. □

Interestingly, if we take the Möbius transformations where $|a| = 1, c = 0, d = 1$ we get $f(z) = az + b, |a| = 1$. This happens to be the complex form of the Euclidean Isometries, hence the group $E_2$ is a subgroup of $Mob(\mathbb{C})$. We now introduce an important invariant of the Möbius group, the cross ratio.Hvidsten (2016)

**Definition 5.7.** The **cross ratio** of four complex numbers $z_0, z_1, z_2, z_3 \in \mathbb{C}$, denoted $(z_0, z_1, z_2, z_3)$ is the value of,

$$\frac{z_0 - z_2}{z_0 - z_3}\frac{z_1 - z_3}{z_1 - z_2}$$

This may seem familiar but slightly different to what appears in definition 5.2. we can indeed show,

**Theorem 5.8.** *The Poincaré distance from two points $z_0, z_1 \in \mathbb{C}$ is given by*

$$\begin{aligned} d(z_0, z_1) &= |\ln((z_0, z_1, w_1, w_0))| \\ &= |\ln(\frac{z_0 - w_1}{z_0 - w_0}\frac{z_1 - w_0}{z_1 - w_1})| \end{aligned}$$

*where $w_0$ and $w_1$ are the points where the hyperbolic line through $z_0$ and $z_1$ meet the boundary $\partial\mathfrak{D}$*

*Proof.* Since for any two $z, w \in \mathbb{C}$, $|zw| = |z||w|$ and $|\frac{z}{w}| = \frac{|z|}{|w|}$, from 5.2 we have,

$$\begin{aligned} d(z_0, z_1) &= |\ln(\frac{|z_0 - w_1||z_1 - w_0|}{|z_0 - z_w||z_1 - w_1|})| \\ &= |\ln(|\frac{z_0 - w_1}{z_0 - w_0}\frac{z_1 - w_0}{z_1 - w_1}|)| \\ &= |\ln(\frac{z_0 - w_1}{z_0 - w_0}\frac{z_1 - w_0}{z_1 - w_1})| \end{aligned}$$

The last line is true since the cross ratio always positive, and a real number if all points lie on a hyperbolic line see Hvidsten (2016). □

**Theorem 5.9.** *If a Möbius transformation has 3 or more fixed points then it is the identity function $id$*

*Proof.* For a Möbius transformation $f(z) = \frac{az+b}{cz+d}$, there are fixed points when $f(z) = z$ so,

$$\frac{az+b}{cz+d} = z \implies cz^2 + (d-a)z - b = 0 \tag{5.1}$$

equation 5.1 has at most two solutions unless $f = id$ then the equation becomes $0 = 0$ and there are infinite solutions. Therefore if there are three or more fixed points $f = id$. □

**Corollary 5.9.1** (Hvidsten (2016)). *Let $z_1, z_2, z_3 \in \mathbb{C}$ be distinct. There exists a unique Möbius transformation, f, such that $f(z_1) = w_1, f(z_2) = w_2, f(z_3) = w_3$ for given $w_1, w_2, w_3 \in \mathbb{C}$*

*Proof.* Let $g_1(z) = (z, z_1, z_2, z_3) = \frac{z-z_2}{z-z_3}\frac{z_1-z_3}{z_1-z_2}$, Then $g_1$ is a Möbius transformation, $g_1$ maps $z_1 \mapsto 1$, $z_2 \mapsto 0$, $z_3 \mapsto \infty$†. Let $g_2(w) = (w, w_1, w_2, w_3) = \frac{w-w_2}{w-w_3}\frac{w_1-w_3}{w_1-w_2}$, similarly $g_2$ is a Möbius transformation which maps, $w_1 \mapsto 1$, $w_2 \mapsto 0$, $w_3 \mapsto \infty$, Hence $f = g_2^{-1} \circ g_1$ maps $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$ and is a Möbius transformation. Suppose there exists $f'$ which also mapped $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$, then $f^{-1} \circ f'$ has at least three fixed points, $z_1, z_2, z_3$ so by 5.9 $f^{-1} \circ f' = id \implies f' = f$. Therefore f is unique.
(†we allow the 'extended complex numbers' to include infinity) □

Now we can prove that the cross ratio is invariant under Möbius transformations.

**Theorem 5.10.** *Let $z_1, z_2, z_3 \in \mathbb{C}$ be distinct and $f$ a Möbius transformation, then for any $z \in \mathbb{C}$*

$$(z, z_1, z_2, z_3) = (f(z), f(z_1), f(z_2), f(z_3))$$

*Proof.* Let $g(z) = (z, z_1, z_2, z_3)$ then $g \circ f^{-1}$ will map $f(z_1) \mapsto 1$, $f(z_2) \mapsto 0$, $f(z_3) \mapsto \infty$. We also have that $h(z) = (z, f(z_1), f(z_2), f(z_3))$ maps $f(z_1) \mapsto 1$, $f(z_2) \mapsto 0$, $f(z_3) \mapsto \infty$, hence by theorem 5.9.1 $h = g \circ f^{-1}$. The result follows since,

$$\begin{aligned}(z, z_1, z_2, z_3) &= g \circ f^{-1}(f(z)) \\ &= h(f(z)) \\ &= (f(z), f(z_1), f(z_2), f(z_3))\end{aligned}$$

□

Now we can prove a useful theorem which helps us to think about how Möbius transformations act on the Poincaré disc.

**Theorem 5.11** (Hvidsten (2016)). *Any Möbius transformation, f, maps circles and lines to circles and lines.*

*Proof.* Let c be a circle or line and let $z_1, z_2, z_3 \in \mathbb{C}$ be distinct points on c. Let $f$ be a Möbius transformation. The points $f(z_1), f(z_2), f(z_3)$ will lie on a line or determine a unique circle we will call c'. Let $z$ be any point on c which is not $z_1, z_2$ or $z_3$. Then the cross product

$$(f(z), f(z_1), f(z_2), f(z_3)) = (z, z_1, z_2, z_3)$$

is real since $z, z_1, z_2, z_3$ all lie on a circle or line Hvidsten (2016) Thm 17.6. Therefore $f(z), f(z_1), f(z_2), f(z_3)$ lie on a circle or line by the same theorem. □

Now finally we can show the main two results of this section.

**Theorem 5.12.** *Any Möbius transformation, $f$, which maps the Poincaré disc, $\mathfrak{D}$ to itself and the boundary $\partial\mathfrak{D}$ to itself has the form,*

$$f(z) = \beta\frac{z-\alpha}{\bar{\alpha}z-1}$$

*with* $|\alpha| < 1$ *and* $|\beta| = 1$

*Proof.* The proof of this theorem uses other ideas from hyperbolic geometry and can be found in Hvidsten (2016). □

Finally, that Möbius transformations are indeed isometries of the Poincaré disc.

**Theorem 5.13.** *Transformations of the form*

$$f(z) = \beta\frac{z-\alpha}{\bar{\alpha}z-1}$$

*Where* $|\alpha| < 1$ *and* $|\beta| = 1$ *are isometries of the Poincaré disc, that is the preserve the Poincaré distance in 5.2.*

*Proof.* Let $f$ be the Möbius transformation described and $z_0, z_1 \in \mathfrak{D}$ be points in the Poincaré disc with $w_0, w_1 \in \partial\mathfrak{D}$ the points where the line through $z_0$ and $z_1$ meets the boundary. Then since $f$ is a Möbius transformation it preserves angles and therefore by theorem 5.11 it will map hyperbolic lines to hyperbolic lines since the right angle with the boundary will be preserved. Therefore $f(z_0), f(z_1)$ are points in $\mathfrak{D}$ and $f(z_0), f(z_1), f(w_0), f(w_1)$ will all lie on the same hyperbolic line. Also since $f$ maps $\partial\mathfrak{D}$ to itself $f(w_0), f(w_1)$ will lie on the boundary. Finally we then have,

$$\begin{aligned} d(f(z_0), f(z_1)) &= |\ln((f(z_0), f(z_1), f(w_1), f(w_0)))| \qquad Theorem \quad 5.8 \\ &= |\ln((z_0, z_1, w_1, w_0))| \\ &= d(z_0, z_1) \end{aligned}$$

Hence $f$ is an isometry of the Poincaré disc. □

We have ended up with a result which shows that all Möbius transformations of a certain form are indeed isometries of the Poincaré disc, however this does not prove the converse statement that all isometries of the Poincaré disc are Möbius transformations of this form. Indeed these turn out to be only the *orientation preserving* isometries, i.e not the reflections. It takes more work to find the complete set of isometries but they can all be written in the form $r \circ g$ where g is a Möbius transformation of the form above.

## 5.4 Fuchsian Groups

Now we have a set of isometries of the hyperbolic plane we need to find what is the equivalent of a wallpaper group in this geometry. As motivation we are looking for we think of tilings of the hyperbolic plane such as in figure 11 and notice they are *finite movements* of the hyperbolic plane. So we want to find *finite* subgroups but also include infinite motions such as a hyperbolic translation so we extend our definition to *discrete* subgroups. Olivares (2018).

**Definition 5.14.** A group G is said to be **discrete** if for a given metric (distance function) $\rho$ such that,

$$\forall g \in G \exists r \in \mathbb{R} \text{ such that } B_r(g) \cap G = g$$

Where $B_r(g) = \{h \in G | \rho(g,h) < r\}$

This is just a fancy way of saying there is a non-zero distance between every element in the group. In this case the distance is not the Poincaré distance but a different abstract distance between elements of $Mob(\mathbb{C})$; this metric defined using the image of the function on a domain, if the images of two different Möbius transformations are sufficiently different then they will have a non-zero 'distance'. We just need to think of definition 5.14 as almost finite groups.

**Definition 5.15.** A **fuchsian** group is a discrete subgroup of $Mob(\mathbb{C})$

The fuchsian groups are equivalent to wallpaper groups. They are almost the most general form of crystallographic groups however they do not include orientation reversing subgroups but for our case this will not matter.

Complete classification of all the fuchsian groups has been studied by initially by Poincaré and then refined by Klein and Coxeter, the theory is very well researched and uses knowledge from topology. The theory extends to all the discrete subgroups of the isomorphism group and not just the orientation preserving subgroups. See Macbeath (1967) for an in depth classification.

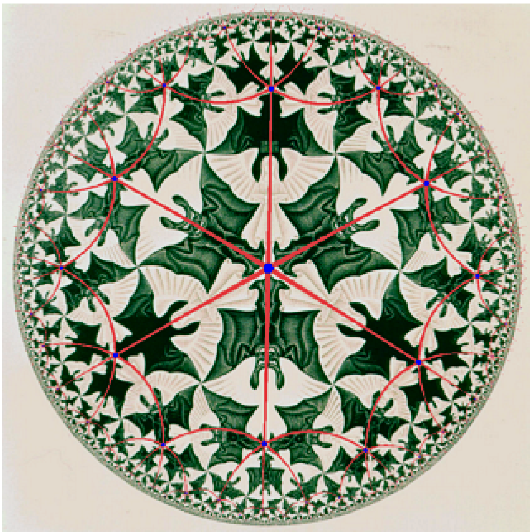


Figure 14: https://en.wikipedia.org/wiki/Order-6_square_tiling

I want to conclude this section with a description of the fuchsian group for figure 14 (https://en.wikipedia.org/wiki/Order-6_square_tiling). Each red quadrilateral in figure 14 is actually a hyperbolic square and each one is congruent in the geometry, these red also represent mirrors. Each blue dot represents a rotation of order 6. In each square there are four half bats, one bat in each square is a darker shade, had this not been the case there would have been a rotation of order 4 at the centre of each square.

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# A Appendix: Proofs

*Proof.* **The semidirect product is a group** Let $(G, \bullet)$ and $(H, *)$ be groups. The semidirect product $G \times_\phi H$ with $\phi : H \to Aut(G)$, $\phi(h) \mapsto \phi_h$ is a group if it satisfies: an identity element, closure, inverses, associativity. Firstly,
Identity:
$(e_g, e_h) \in G \times_\phi H$, for any $(g, h) \in G \times_\phi H$,

$$\begin{aligned}(g,h)(e_g,e_h) &= (g \bullet \phi_h(e_h), h * e_h) \\ &= (g \bullet e_g, h) && \textit{since } \phi_h \textit{ is an automorphism} \\ &= (g,h)\end{aligned}$$

and

$$\begin{aligned}(e_g,e_h)(g,h) &= (e_g \bullet \phi_{e_h}(g), e_h * h) \\ &= (e_g \bullet g, h) && \textit{since } \phi \textit{ is a homomorphism, so } \phi_{e_h} \textit{ is the identity of Aut(G)} \\ &= (g,h)\end{aligned}$$

So $(e_g, e_h)$ is the identity of $G \times_\phi H$.
Closure:
For any $(g, h), (g', h') \in G \times_\phi H$,

$$(g,h)(g',h') = (g \bullet \phi_h(g'), h * h') \in G \times_\phi H$$

Since $g \bullet \phi_h(g') \in G$ and $h * h' \in H$.
Inverse:
For each $(g, h) \in G \times_\phi H$ there is a unique $(\phi_{h^{-1}}(g^{-1}), h^{-1}) \in G \times_\phi H$. Then we have,

$$\begin{aligned}(g,h)(\phi_{h^{-1}}(g^{-1}), h^{-1}) &= (g \bullet \phi_h(\phi_{h^{-1}}(g^{-1})), h * h^{-1}) \\ &= (g \bullet \phi_{h*h^{-1}}(g^{-1}), e_h) && \textit{since } \phi \textit{ is a homomorphism} \\ &= (g \bullet g^{-1}, e_h) \\ &= (e_g, e_h)\end{aligned}$$

and

$$\begin{aligned}(\phi_{h^{-1}}(g^{-1}), h^{-1})(g,h) &= (\phi_{h^{-1}}(g^{-1})\phi_{h^{-1}}(g), h^{-1} * h) \\ &= ((\phi_{h^{-1}}(g))^{-1}\phi_{h^{-1}}(g), e_h) \qquad \textit{since } \phi_{h^{-1}} \textit{ is a homomorphism} \\ &= (e_g, e_h)\end{aligned}$$

so each element $(g, h)$ has unique inverse $(\phi_{h^{-1}}(g^{-1}), h^{-1})$.
Associativity:
For any $(g, h), (g', h'), (g'', h'') \in G \times_\phi H$ we have,

$$\begin{aligned}(g,h)[(g',h')(g'',h'')] &= (g,h)(g' \bullet \phi_{h'}(g''), h' * h'') \\ &= (g \bullet \phi_h(g' \bullet \phi_{h'}(g'')), h * h' * h'') \\ &= (g \bullet \phi_h(g') \bullet \phi_h(\phi_{h'}(g'')), h * h' * h'') \quad \textit{since } \phi_h \textit{ is a homomorphism} \\ &= (g \bullet \phi_h(g') \bullet \phi_{h*h'}(g''), h * h' * h'') \quad \textit{since } \phi \textit{ is a homomorphism} \\ &= (g \bullet \phi_h(g'), h * h')(g'', h'') \\ &= [(g,h)(g',h')](g'',h'')\end{aligned}$$

Hence the semidirect product is associative and therefore a group. □

*Proof.* **Theorem 2.6** Assume $E$ is a group with $H, J$ subgroups such that $H$ is normal in $E$, $E = GH$ and $G \cap H = \{e\}$. Define the map

$$\begin{aligned}\psi : G \times_\phi H &\to E \\ (g,h) &\mapsto gh\end{aligned}$$

with

$$\begin{aligned}\phi : H &\to Aut(G) \\ \phi(h)(g) &\mapsto \phi_h(g) \mapsto hgh^{-1}\end{aligned}$$

Then $\psi$ is well defined since for any $(x, y)$ and $(x', y')$ where $(x, y) = (x', y')$ we have $x = x'$ and $y = y'$ so $xy = x'y'$. $\psi$ is a homomorphism since for all $(g, h)$ and $(g', h')$:

$$\begin{aligned}\psi((g,h),(g',h')) &= \psi(g\phi(h)(g'), hh') \\ &= \psi(ghg'h^{-1}, hh') \\ &= ghg'h^{-1}hh' \\ &= ghg'h' \\ &= \psi(g,h)\psi(g',h').\end{aligned}$$

The image of $G \times_\phi H$ is $E$ since $E = GH$ so every element $f \in E$ can be written as a product $f = gh$ for $g \in G$ and $h \in H$, Hence $\psi$ is surjective. Assume for any $f \in E$, $f$ can be written two ways, $f = gh = g'h'$ then

$$\begin{aligned}&gh = g'h' \\ &\implies (h')^{-1}h = g'g^{-1} \\ &\implies (h')^{-1}h \in G \cap H \text{ and } g'g^{-1} \in G \cap H \\ &\implies (h')^{-1}h = g'g^{-1} = e \\ &\implies h = h' \text{ and } g = g'\end{aligned}$$

so each $f$ is written uniquely as a product $f = gh$ and thus $\psi$ is injective. $\psi$ is therefore and isomorphism. □