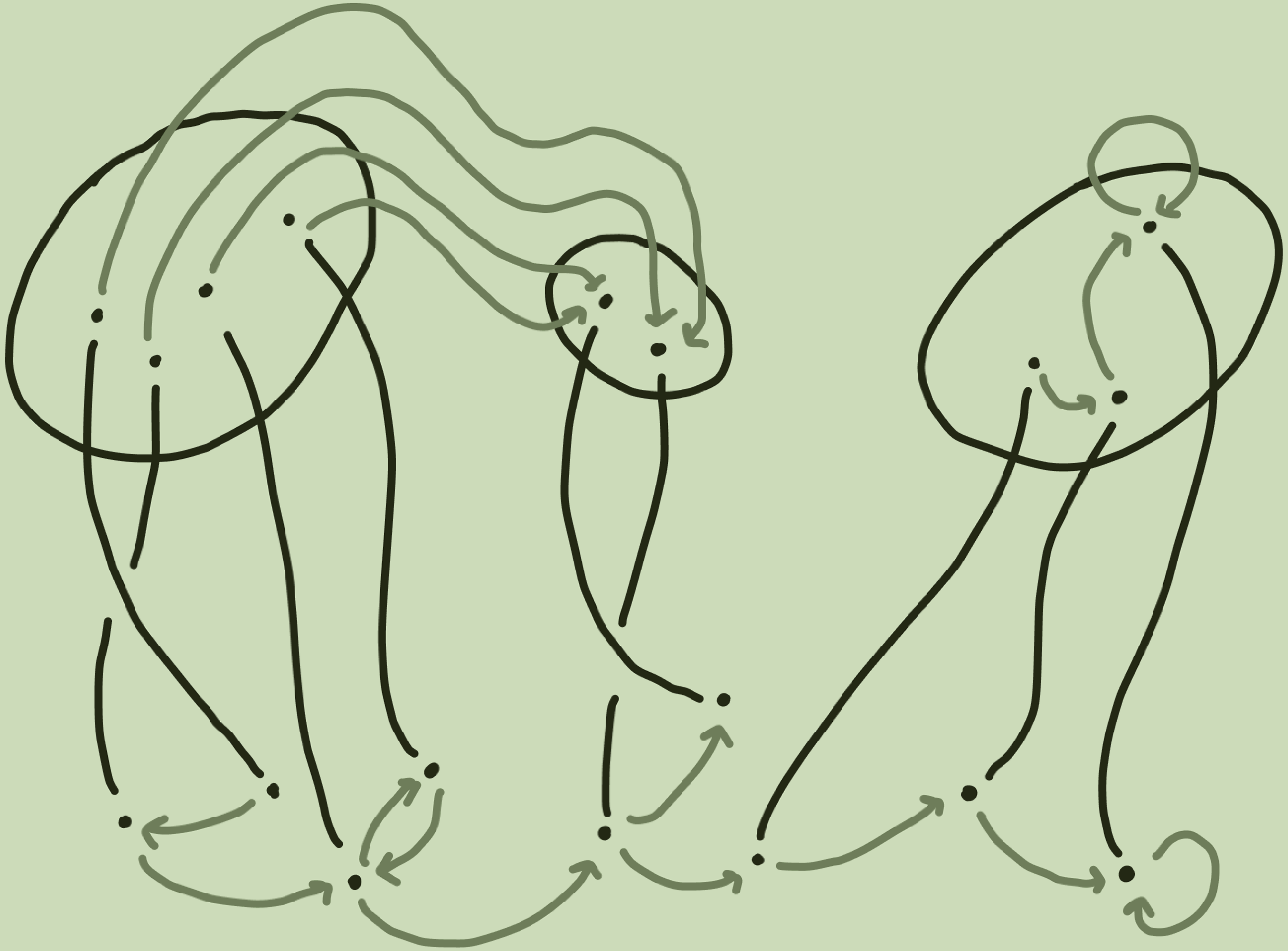
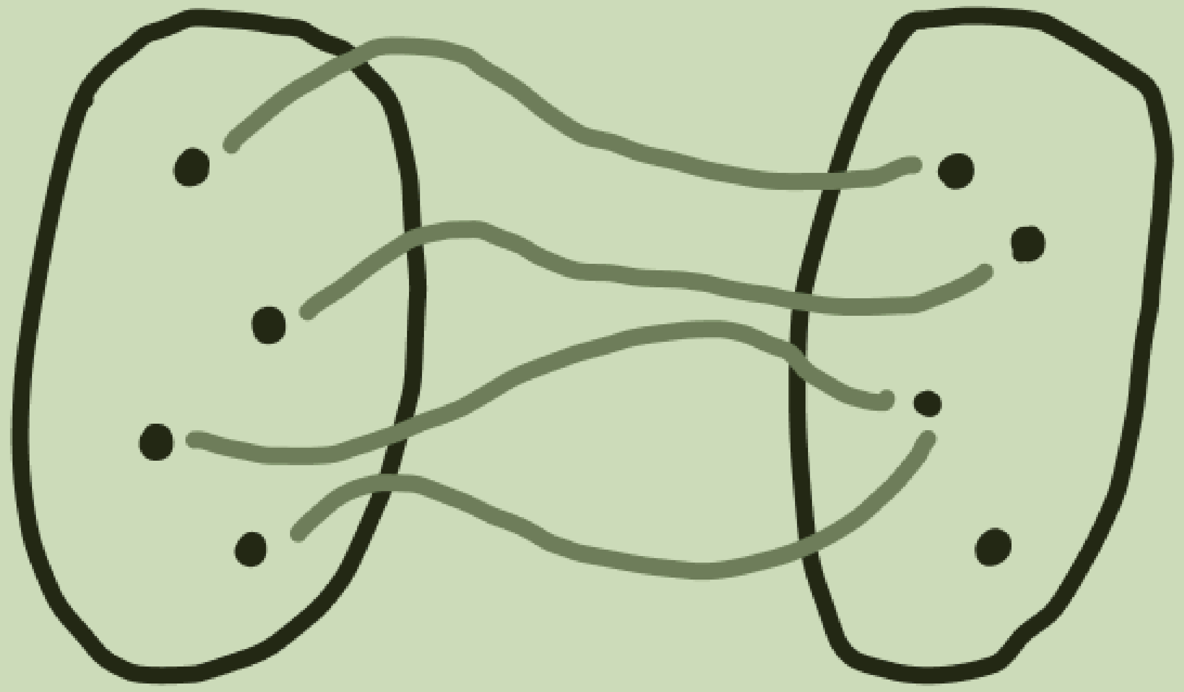
a generalised family construction for virtual double categories

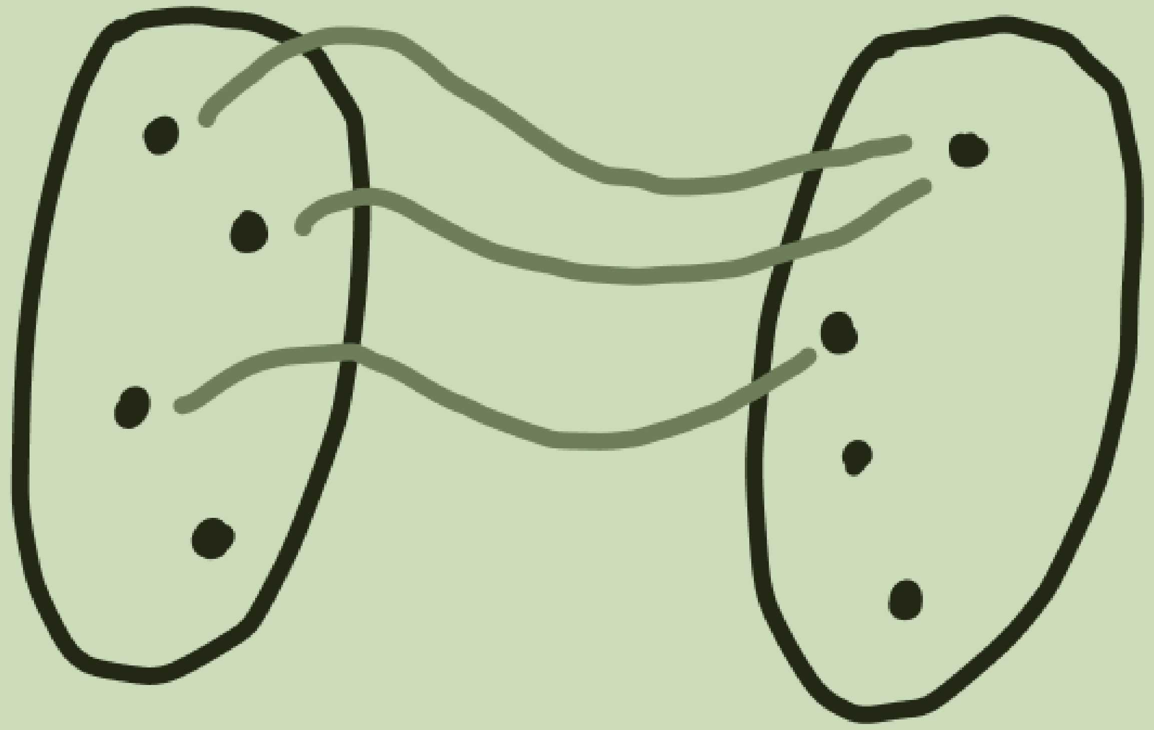


joint work with Evan Patterson

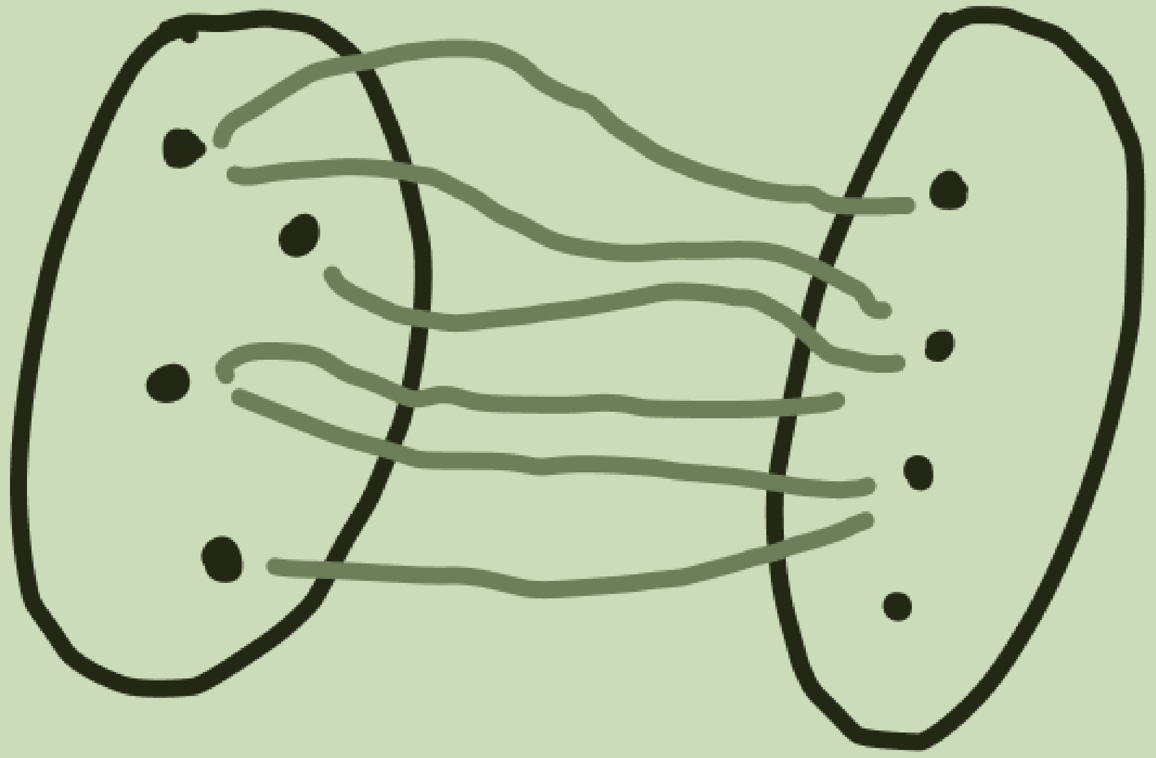
function



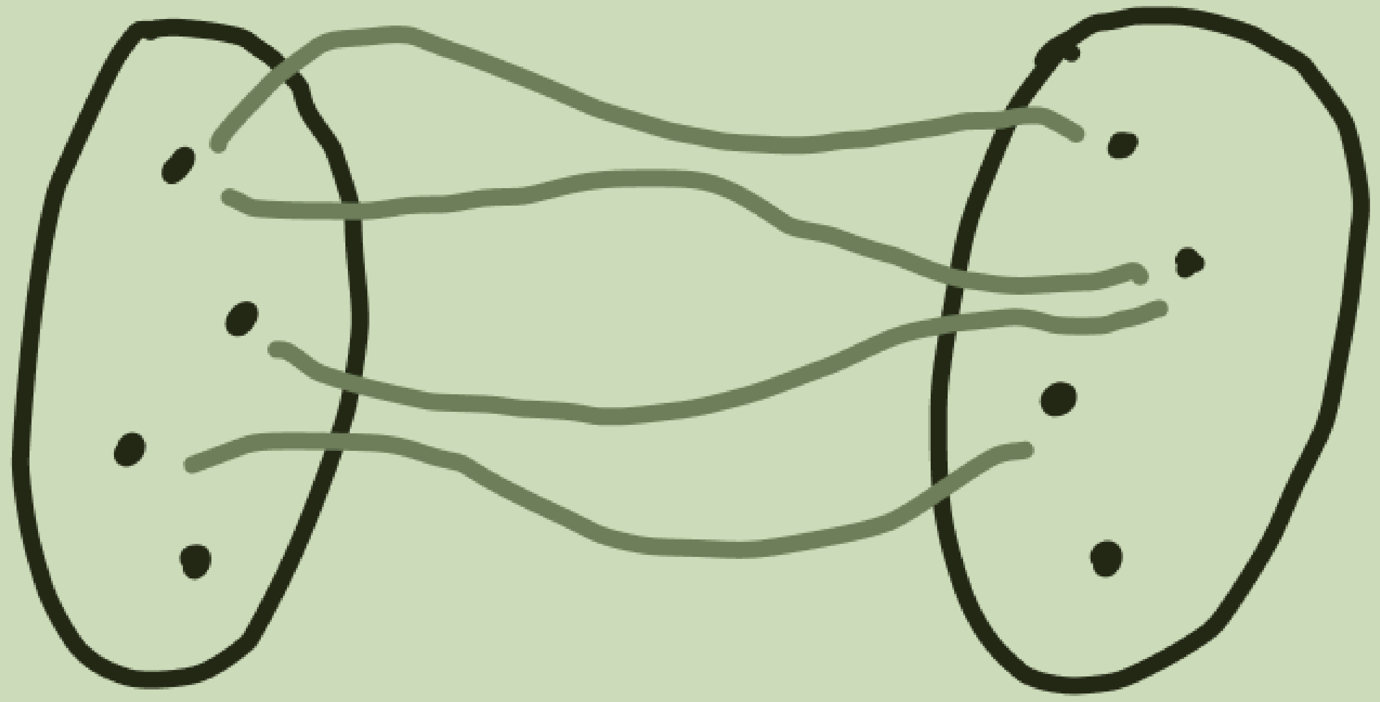
partial function



multivalued function



multivalued partial function



Par

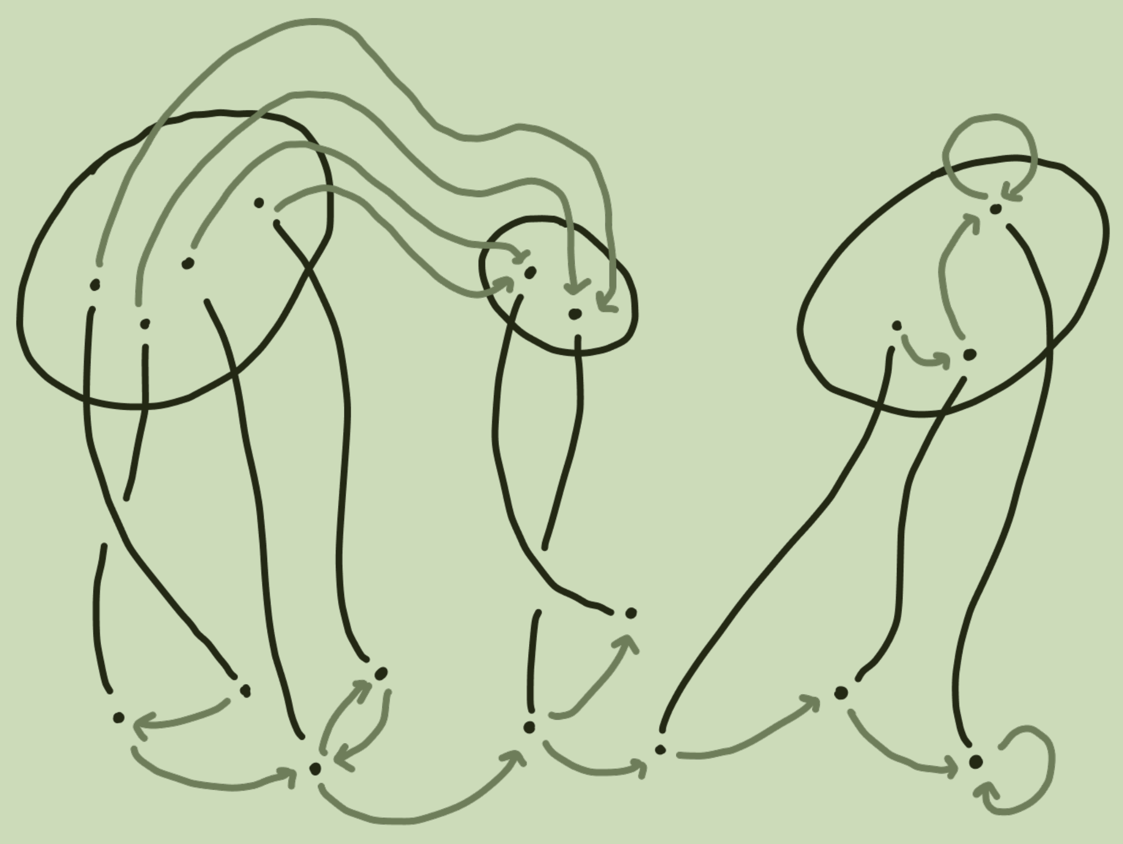
double category of sets, functions and partial functions

Mult

double category of sets, functions and multivalued functions

outline

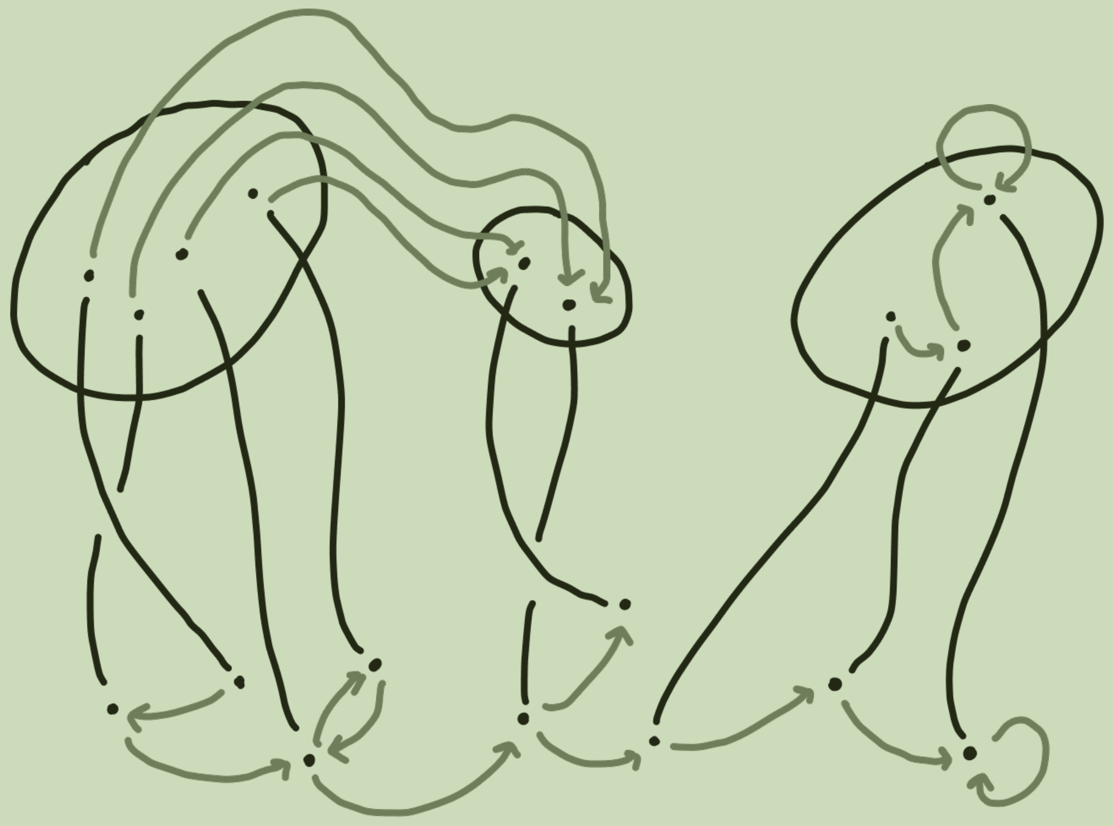
the category of families



for a category $\mathcal{C}$, $\mathrm{Fam}(\mathcal{C})$ is defined:

- objects $(I, \underline{x})$ where I is a Set and $\underline{x}$ is an I-indexed family of objects in $\mathcal{C}$;
- arrows $(f_0, f) : (I, \underline{x}) \longrightarrow (J, \underline{y})$ $f_0 : I \to J$ a function and $f_i : x_i \longrightarrow y_{f_0(i)}$ arrows in $\mathcal{C}$.

the category of families



for a category $\mathcal{C}$, $\mathrm{Fam}^*(\mathcal{C})$ is defined:

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- arrows $(f_0, f) : (I, \underline{x}) \to (J, \underline{y})$ $I \leftarrow J : f_0$ a function and $f_j : x_{f_0(j)} \to y_j$ arrows in $\mathcal{C}$.

$$\begin{tikzcd} & \mathrm{Fam}(\mathcal{C}) \arrow[d, dashed, "\hat{F}"] \\ \mathcal{C} \arrow[ur, "\Delta"] \arrow[r, "F"'] & E \end{tikzcd}$$

E has coproducts,
exists a unique (up to iso)
$\hat{F} : \mathrm{Fam}(\mathcal{C}) \longrightarrow E$
preserving coproducts

the family construction is the free coproduct completion.

the coproduct of families is inherited from Set.

$$\Sigma_! : \mathrm{Fam}^2(\mathcal{C}) \longrightarrow \mathrm{Fam}(\mathcal{C})$$

$$\begin{tikzcd}
(I, (A, \underline{x})) \arrow[r, mapsto] \arrow[d, "(f_0, f_a)_i"'] & \left(\coprod_{i \in I} A_i, \underline{x}\right) \arrow[d, "([f_{0_i}], f_{a_i})"] \\
(J, (B, \underline{y})) \arrow[r, mapsto] & \left(\coprod_{j \in J} B_j, \underline{y}\right)
\end{tikzcd}$$

$$\mathrm{Fam}(1) \cong \mathrm{Set}$$

$1 : \quad \bullet \circlearrowright^{id}$

families → coproducts

$\mathcal{C}$ has coproducts if and only if $\Delta : \mathcal{C} \hookrightarrow \mathrm{Fam}(\mathcal{C})$ has a left adjoint, $\Sigma \dashv \Delta$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\perp"] & \mathrm{Fam}(\mathcal{C}) \arrow[l, bend right, "\Sigma"'] \end{tikzcd}$$

objects: x

composition and units of arrows

arrows: $x \to y$

$$(a \to b \to c) \to d = a \to (b \to c \to d)$$

proarrows: $x \mathrel{\mapstochar\rightarrow} y$

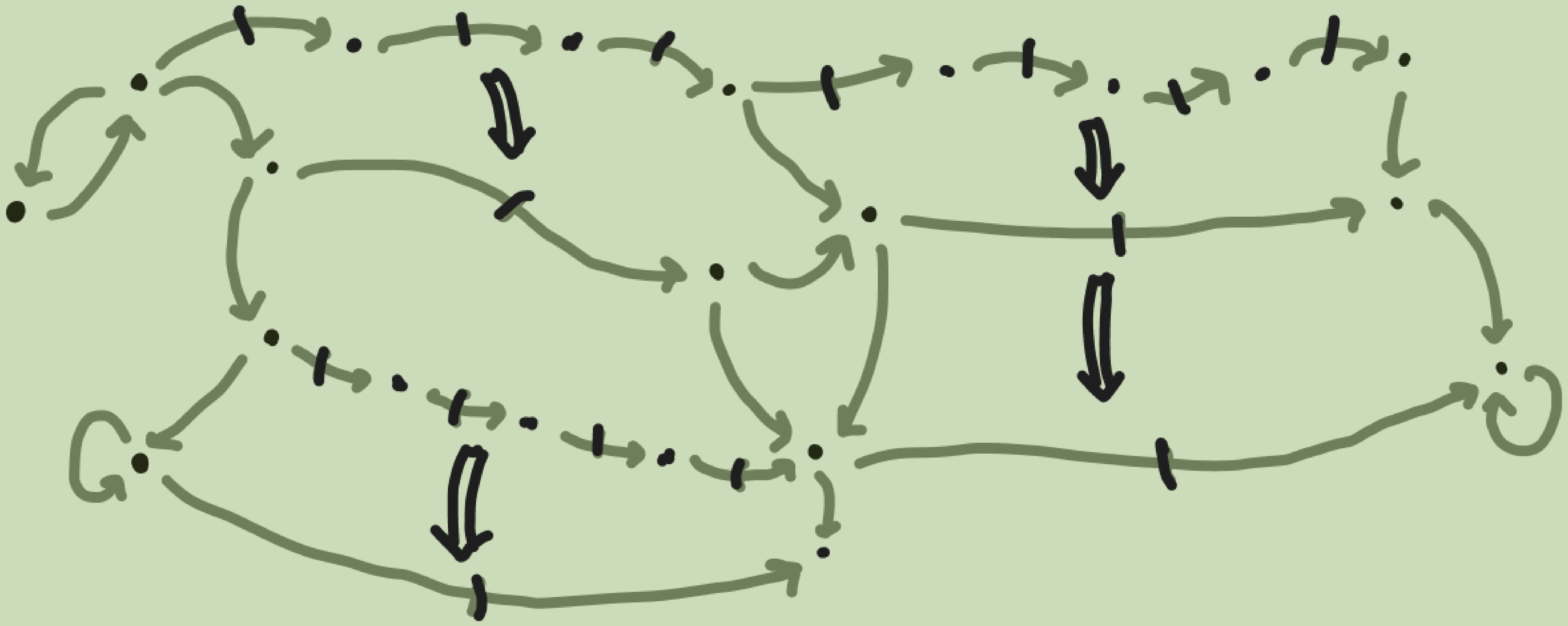
weak composition and units of proarrows

cells:

$$\begin{tikzcd} x \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & y \arrow[d] \\ x' \arrow[r, "|" marking] & y' \end{tikzcd}$$

$$(a \mathrel{\mapstochar\rightarrow} b \mathrel{\mapstochar\rightarrow} c) \mathrel{\mapstochar\rightarrow} d \cong a \mathrel{\mapstochar\rightarrow} (b \mathrel{\mapstochar\rightarrow} c \mathrel{\mapstochar\rightarrow} d)$$

(pseudo) double categories

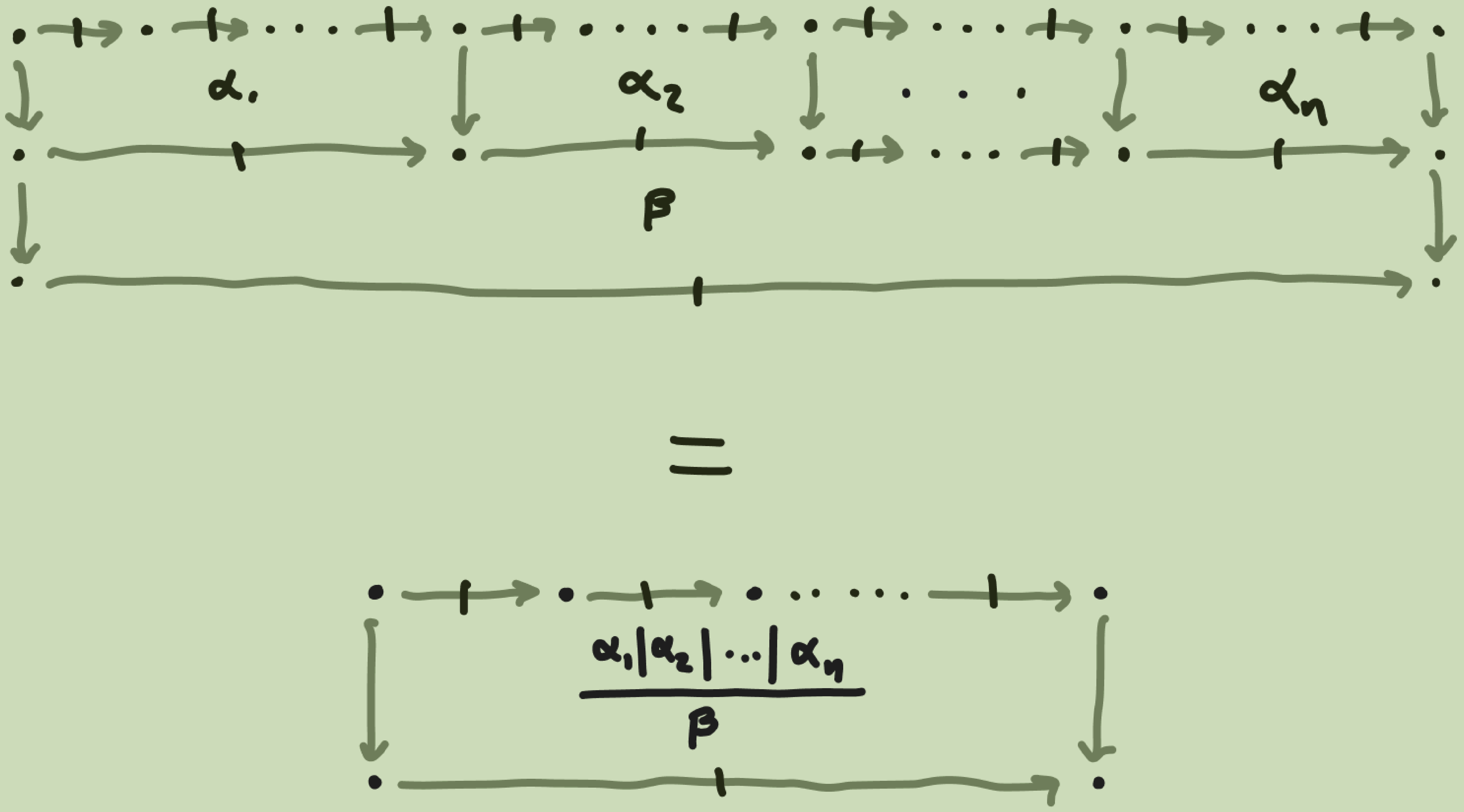


virtual double categories (VDCs)

objects, arrows, proarrows and multicells

[CS10] [Lei04]

cell composition in a virtual double category



virtual double categories generalise double categories. they are also a good place to study formal category theory.

double categories as VDCs

every (pseudo) double category, $\mathbb{D}$ can be viwed as a virtual double category, in fact,

$$\mathrm{Dbl}_{\mathrm{lax}} \hookrightarrow \mathrm{vDbl}$$

is an embedding fully faithful on 1-cells and 2-cells.

example: categories of monoids

for a monoidal category, $(\mathcal{C}, \otimes, 1)$,
we can define the category of monoids
internal to $\mathcal{C}$ as $\mathrm{Mon}(\mathcal{C})$.

$\mathrm{Mon}(\mathcal{C})$ is a monoidal category
only if $\mathcal{C}$ is braided.

[JS86]

monoids internal to a VDC.

a monoid internal to $\mathbb{D}$, (X_0, x, μ, η) consists of:

- an object, X_0
- a proarrow, $x : X_0 \to X_0$
- a multiplication cell,
- a unit cell,

$$\begin{tikzcd} X_0 \arrow[r, "|" marking] \arrow[d, equal] & X_0 \arrow[r, "|" marking] \arrow[d, phantom, "\Downarrow\mu"] & X_0 \arrow[d, equal] \\ X \arrow[rr, "|" marking] & & X_0 \end{tikzcd}$$

$$\begin{tikzcd} & X_0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\Downarrow\eta"] & \\ X_0 \arrow[rr, "|" marking] & {} & X_0 \end{tikzcd}$$

- associativity + unitality axioms!

[CS10]

example: virtual double category of monoids

for a virtual double category $\mathbb{D}$,
we can define a *virtual double category* of monoids $\mathrm{Mon}(\mathbb{D})$.

family construction in virtual double categories, proarrows

$\mathbb{F}\mathrm{am}^*(\mathbb{D}) :=$ $\mathbb{F}\mathrm{am}^*(\mathbb{D})_0 = \mathrm{Fam}^*(\mathbb{D}_0)$

$$\begin{tikzcd} I \arrow[d] & A \arrow[l, "l"'] \arrow[r, "r"] \arrow[d] & J \arrow[d] \\ \mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathbb{D}_0 \end{tikzcd}$$

- proarrows $(A, \underline{m}) : (I, \underline{x}) \longrightarrow\!\!\!\!\!\!\!\!\!\!\shortmid\;\;\;\;\;\; (J, \underline{y})$

are families of proarrows $m_a : x_{l(a)} \longrightarrow\!\!\!\!\!\!\!\!\!\!\shortmid\;\;\;\;\;\; y_{r(a)}$

indexed by a span $I \xleftarrow{l} A \xrightarrow{r} J$;

[Pat26a]

families of cells

$\mathbb{F}\mathrm{am}^*(\mathbb{D}) :=$

cells are

$$\begin{tikzcd}
(I_0, x_0) \arrow[r, "{(A_1, m_1)}", "|" marking] \arrow[d, "{(f, f_0)}"'] & (I_1, x_1) \arrow[r, "{(A_2, m_2)}", "|" marking] & \cdots \arrow[r, "{(A_n, m_n)}", "|" marking] & (I_n, x_n) \arrow[d, "{(g, g_0)}"] \\
(J_0, y_0) \arrow[rrr, "{(B, n)}"', "|" marking] \arrow[urrr, phantom, "\alpha"] & & & (J_1, y_1)
\end{tikzcd}$$

families of cells

$$\begin{tikzcd}
(x_0)_{f_0(j_0)} \arrow[r, "(m_1)_{a_1}", "|" marking] \arrow[d, "f_{j_0}"'] & (x_1)_{i_1} \arrow[r, "(m_2)_{a_2}", "|" marking] & \cdots \arrow[r, "(m_n)_{a_n}", "|" marking] & (x_n)_{g_0(j_1)} \arrow[d, "g_{j_1}"] \\
(y_0)_{j_0} \arrow[rrr, "n_b"', "|" marking] \arrow[urrr, phantom, "\alpha_b"] & & & (y_1)_{j_1}
\end{tikzcd}$$

indexed by a morphism of spans.

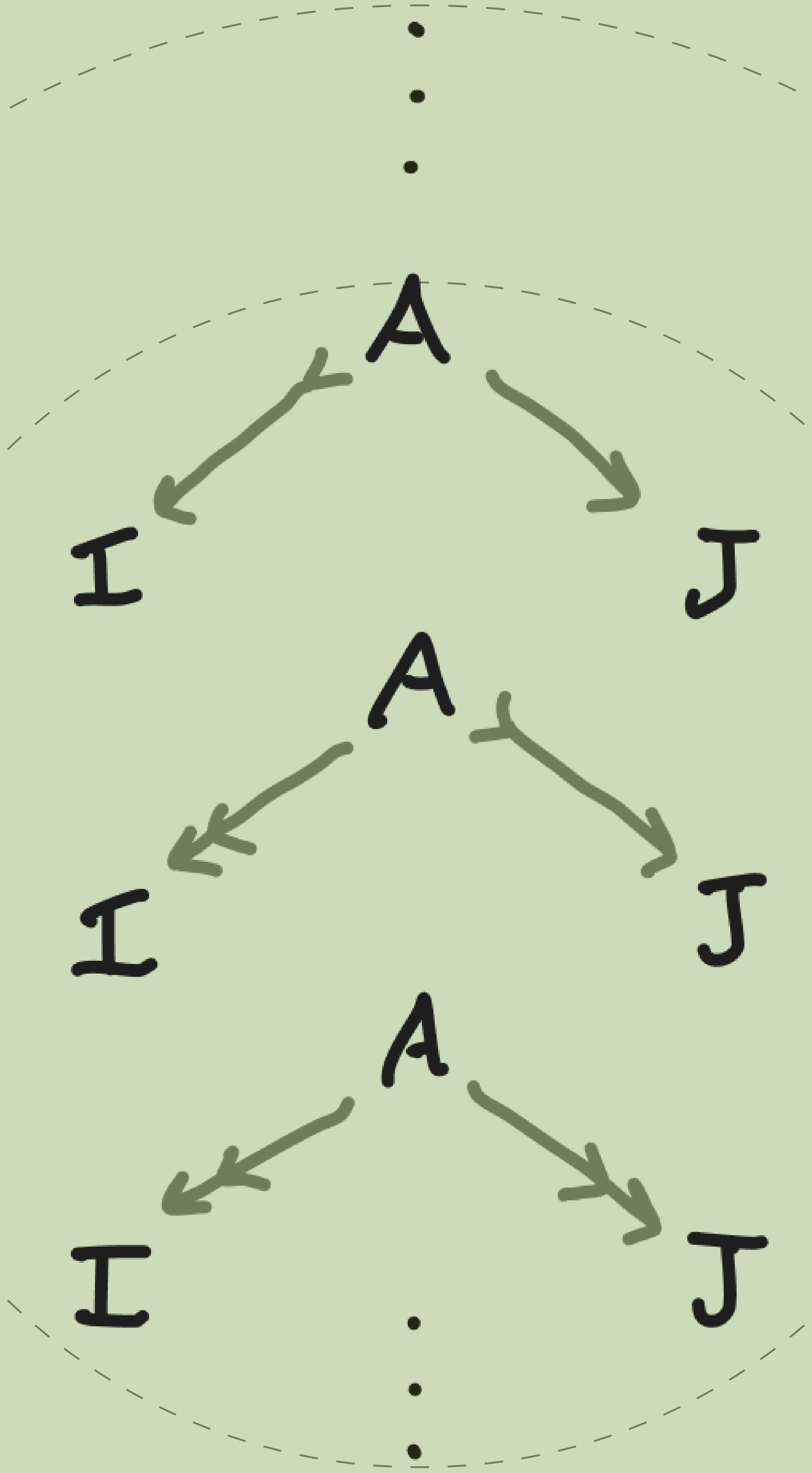
$$\begin{tikzcd}
I_0 & A_1 \times_{I_1} A_2 \times_{I_2} \cdots \times_{I_{n-1}} A_n \arrow[l] \arrow[r] & I_n \\
J_0 \arrow[u, "f_0"] & B \arrow[l] \arrow[r] \arrow[u, "\alpha_0"'] & J_1 \arrow[u, "g_0"']
\end{tikzcd}$$

$b : j_0 \xrightarrow{|} j_1 \qquad \alpha_0(b) = (a_1, a_2, \dots, a_n)$

$$\mathbb{F}\mathrm{am}(\mathbb{1}) \cong \mathbb{S}\mathrm{pan}$$

$$\mathbb{1}: \quad \begin{tikzcd} \bullet \arrow[r, "1", "|"{marking}] \arrow[d, "1"'] \arrow[dr, phantom, "1"] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "1"', "|"{marking}] & \bullet \end{tikzcd}$$

we can choose to index by certain classes of functions to give different kinds of product



span indexing pair

(L, R), $\quad L, R \subseteq \mathrm{Mor}(\mathrm{sk}(\mathrm{Set}))$

(i) pullback stable:

$$\begin{tikzcd} \bullet \arrow[r, tail, "\iota'"] \arrow[d, two heads, "r'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, two heads, "r"] \\ \bullet \arrow[r, tail, "\iota"'] & \bullet \end{tikzcd}$$

(ii) coproduct stable:

$$\begin{tikzcd} & U_1 \sqcup U_2 & \\ U_1 \arrow[ur] & X_1 \sqcup X_2 \arrow[u, tail, dashed] & U_2 \arrow[ul] \\ X_1 \arrow[u] \arrow[ur] & & X_2 \arrow[ul] \arrow[u] \end{tikzcd}$$

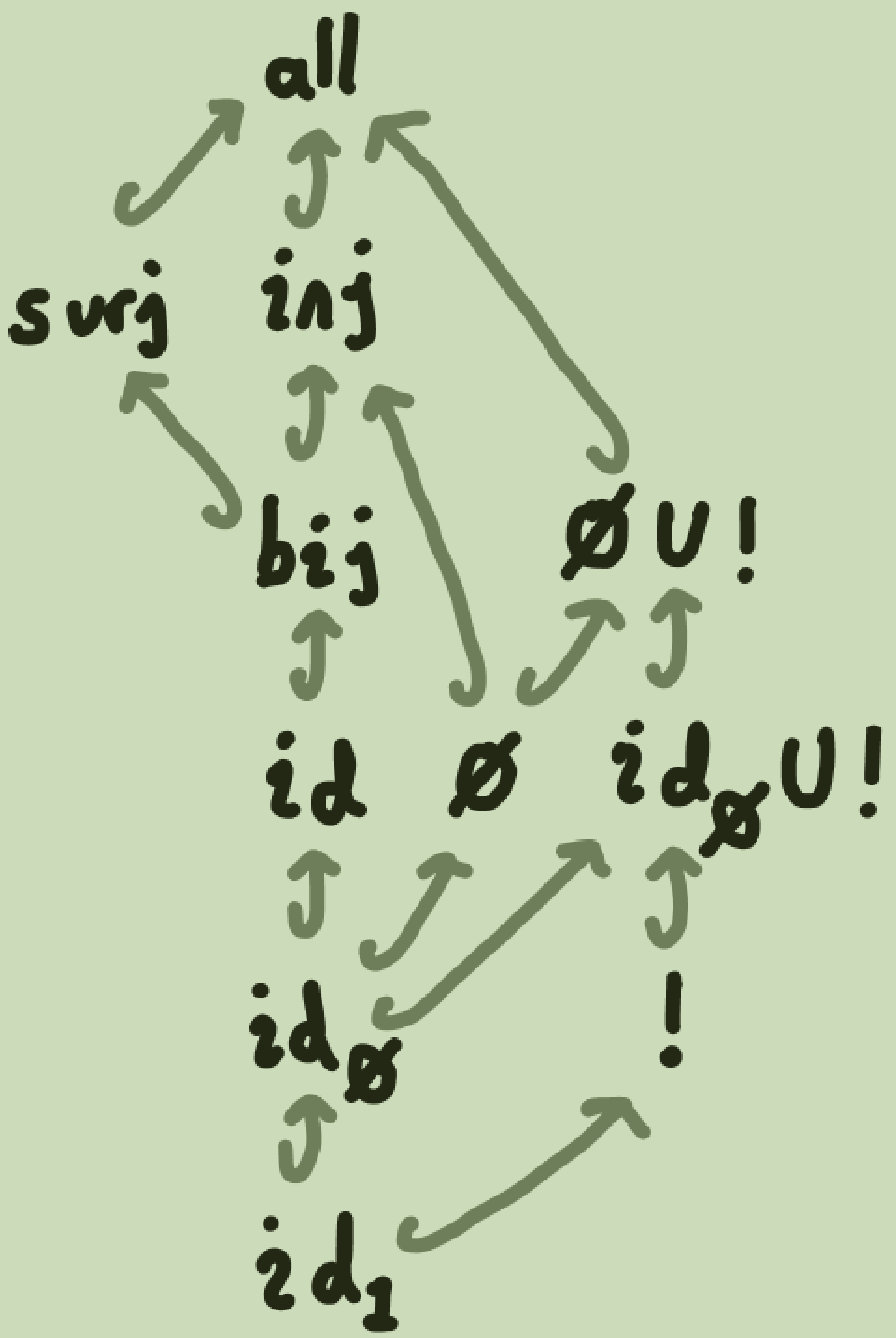
$I \rightarrowtail J$

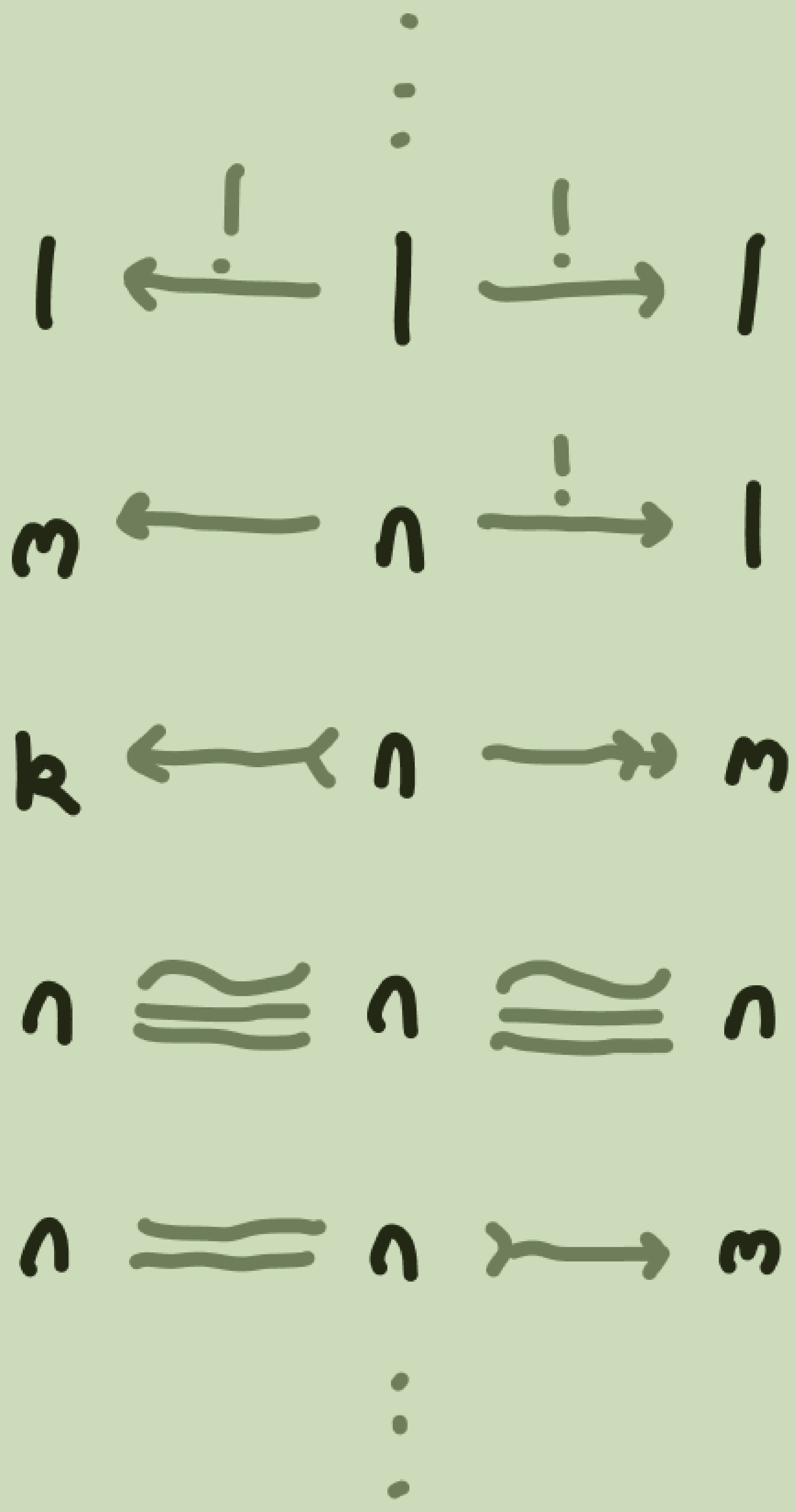
(iii) unit:

$1 \rightarrowtail\!\!\twoheadrightarrow 1$

examples: span indexing schemes

$L, R \in \{\text{all},$
$\text{surj}, \text{inj},$
$\text{bij}, \text{id}, !, \emptyset,$
$\text{id}_\emptyset := \{\text{id}_\emptyset, \text{id}_1\}, \text{id}_1\}$





(L,R)-family construction in virtual double categories

$\mathbb{F}\mathrm{am}^*(\mathbb{D})_{LR}$

- $\mathbb{F}\mathrm{am}^*(\mathbb{D})_{LR_0} = \mathrm{Fam}^*(\mathbb{D}_0)$
- proarrows $(A,\underline{m}) : (I,\underline{x}) \xrightarrow{\shortmid} (J,\underline{y})$ are families of proarrows $m_a : x_{l(a)} \xrightarrow{\shortmid} y_{r(a)}$ indexed by a span $I \xleftarrow{l} A \xrightarrow{r} J$, $l \in L$ $r \in R$;
- cells are families of cells

$$\begin{tikzcd}
x_0 \arrow[r, "\shortmid" marking] \arrow[d, "f"'] & x_1 \arrow[r, "\shortmid" marking] & \cdots \arrow[r, "\shortmid" marking] & x_n \arrow[d, "g"] \\
y_0 \arrow[rrr, "\shortmid" marking] \arrow[urrr, phantom, "\alpha_b"] & & & y_1
\end{tikzcd}$$

indexed by a morphism of spans.

Thm

$\mathbb{F}\mathrm{am}^{(*)}(-)_{L,R} : v\mathbb{D}\mathrm{bl} \longrightarrow v\mathbb{D}\mathrm{bl}$ is a (co)lax-idempotent pseudomonad.

colax-idempotent pseudomonads

$$\begin{tikzcd} \mathbb{D} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathbb{F}am^*(\mathbb{D})_{L,R} \arrow[l, bend left, "\Pi"] \end{tikzcd}$$

Π is the monad multiplication, it is also right adjoint to the unit of the monad Δ.

[KL97]

families in double categories

when $\mathbb{D}$ is a pseudo double category

$\mathbb{F}\mathrm{am}(\mathbb{D})_{LR}$ is a pseudo double category

hence $\mathbb{F}\mathrm{am}(-)_{LR}$ restricts to a monad

on Dbl_{lax}

$$\mathbb{F}\mathrm{am}_{(\mathrm{inj},\mathrm{all})}(\mathbb{1}) \cong \mathbb{P}\mathrm{ar}$$

$$\mathbb{1}: \quad \begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "1"] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "1"'] & \bullet \end{tikzcd}$$

$$\mathbb{F}am_{(Surj,all)}(\mathbb{1}) \cong \mathbb{M}ult$$

$\mathbb{1}$:
$$\begin{tikzcd}
\bullet \arrow[r, "1", "|" marking] \arrow[d, "1"'] \arrow[dr, phantom, "1"] & \bullet \arrow[d, "1"] \\
\bullet \arrow[r, "1"', "|" marking] & \bullet
\end{tikzcd}$$

definition: (L,R)-products

if $\Delta : \mathbb{D} \longrightarrow \mathbb{F}\mathrm{am}^*(\mathbb{D})_{L,R}$ has a right adjoint, Π

we say $\mathbb{D}$ has (L,R)-products.

- $\prod_{i \in I} x_i := \Pi(I, \underline{x})$;
- $\prod_{a \in A} m_a := \Pi(A, \underline{m})$.

example: parallel products; (id,id)-products

index by

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, equal] & \\ I & & I \end{tikzcd}$$

$$\begin{array}{c} x_1 \overset{p_1}{\longrightarrow} y_1 \\ \vdots \\ x_i \overset{p_i}{\longrightarrow} y_i \\ \vdots \end{array} \quad \overset{\Pi}{\rightsquigarrow} \quad \prod_{i \in I} x_i \xrightarrow{\prod_{i \in I} p_i} \prod_{i \in I} y_i$$

in Span:

$$\left(x \xleftarrow{f} s \xrightarrow{g} y \right) \times \left(x' \xleftarrow{f'} s' \xrightarrow{g'} y' \right)$$

$$=$$

$$x \times x' \xleftarrow{\langle f\pi_s, f'\pi_{s'} \rangle} s \times s' \xrightarrow{\langle g\pi_s, g'\pi_{s'} \rangle} y \times y'$$

example: prodiagonals

in a unitial virtual double category $\mathbb{D}$,

$$\Pi\left(1 \xleftarrow{!} 2 \overset{\mathrm{id}}{=\!=} 2,\ x \begin{matrix} \overset{1}{\nrightarrow} x \\ \underset{1}{\nrightarrow} x \end{matrix} \right) = \delta_x : x \nrightarrow x^2$$

if Π is normal i.e. $1_{\Pi x} \cong \Pi 1_x$ then δ_x is the companion of $\Delta_x : x \to x^2$

in $\mathbb{S}\mathrm{pan}$:

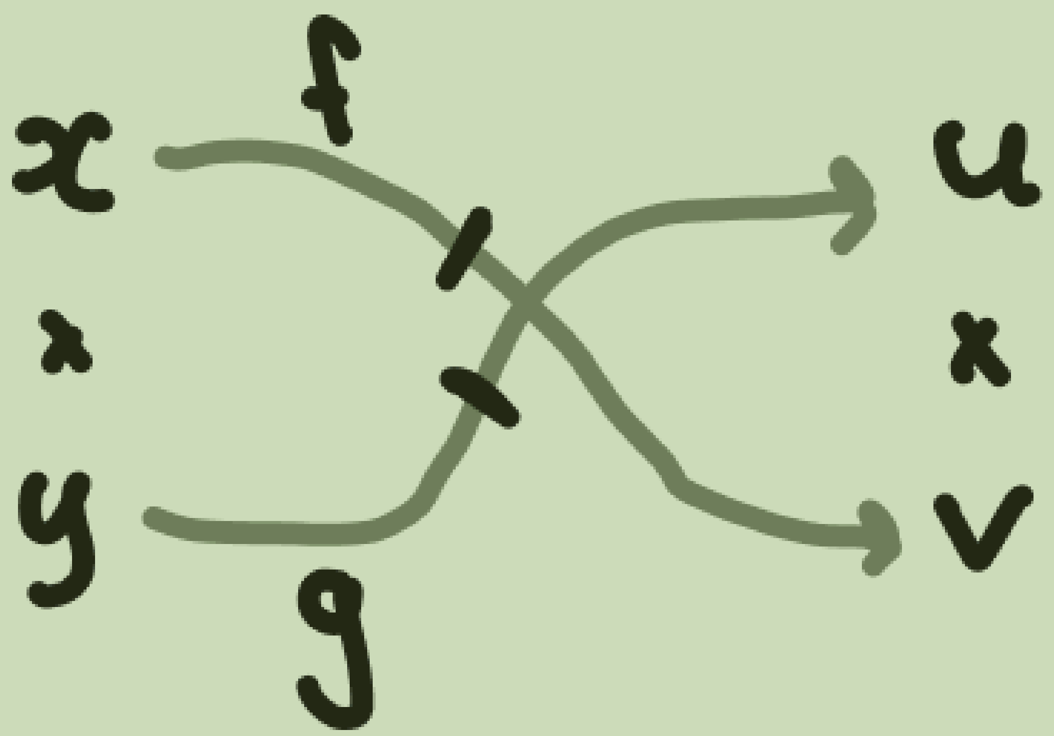
$$\delta_x = x \overset{\mathrm{id}}{=\!=} x \xrightarrow{\Delta x} x^2$$

in $\mathbb{P}\mathrm{rof}$:

$$\delta_{\mathcal{C}} : (\mathcal{C} \times \mathcal{C})^{op} \times \mathcal{C} \to \mathbf{Set}$$

$$((x,y),z) \longmapsto \hom((x,y),(z,z))$$

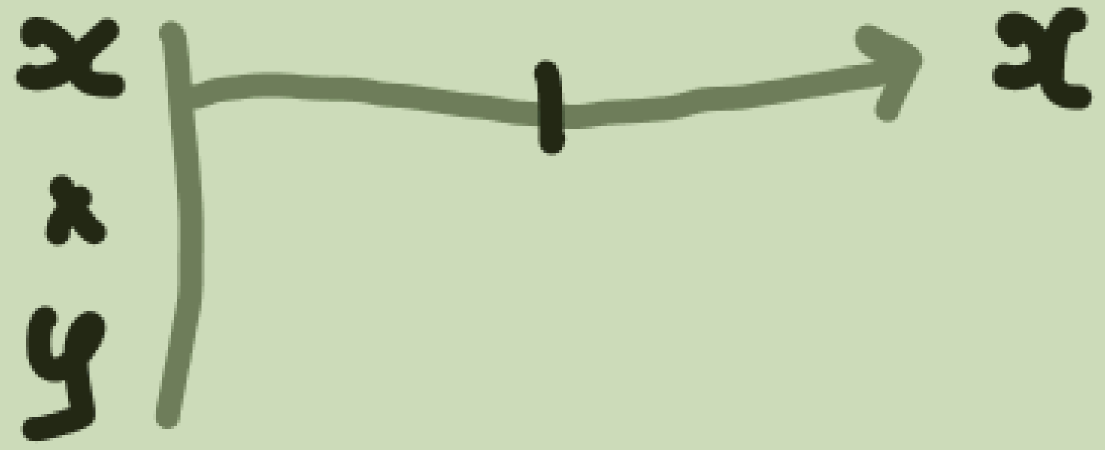
example: other products



braiding

$$2 = 2 \xrightarrow{\sigma_{12}} 2$$

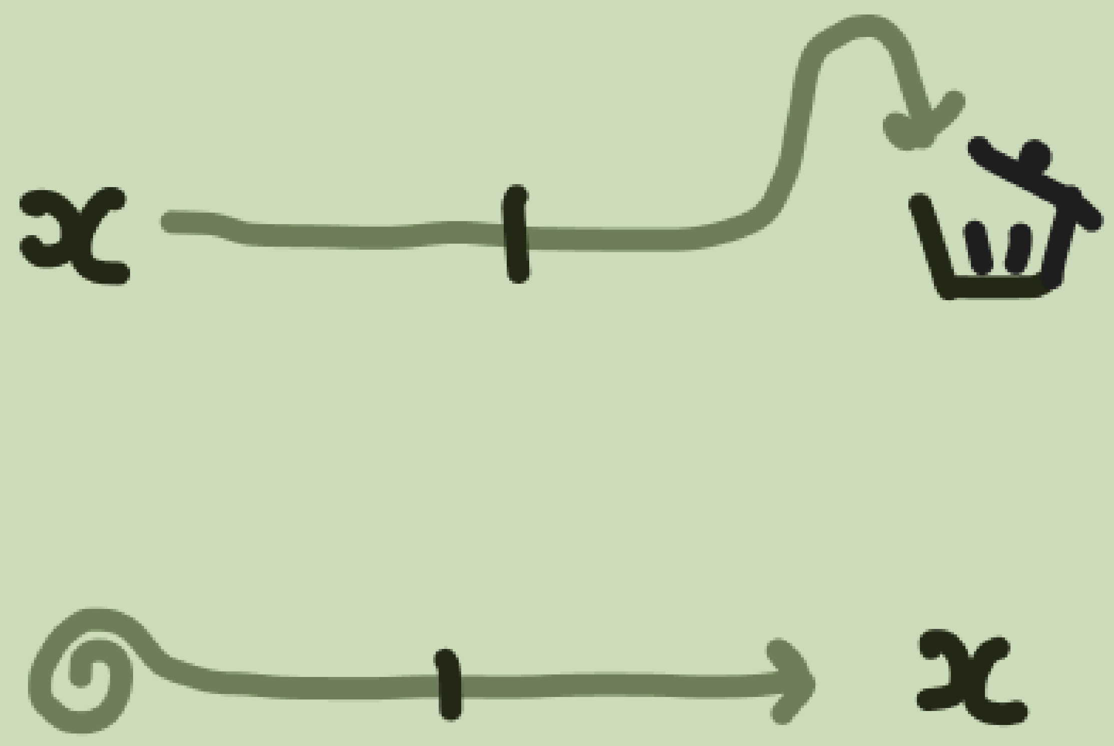
$$\sigma_{fg} : x \times y \mapsto v \times u$$



projection

$$2 \xleftarrow{1} 1 = 1$$

$$\pi_1 : x \times y \mapsto x$$



deletion/creation

$$1 \xleftarrow{!} 0 = 0 \quad \Big| \quad 0 = 0 \xrightarrow{!} 1$$

$$\varepsilon : x \mapsto 1 \quad \Big| \quad \varepsilon^* : 1 \mapsto x$$

binary & nullary products $\overset{\text{generate}}{\rightsquigarrow}$ finite n-fold products

$X \times X \ , \ 1 \rightsquigarrow X \times X \times \cdots \times X$

induct on
$X \times (X \times \cdots \times X)$

generation theorems

in a virtual double category;

binary & nullary parallel products $\overset{\text{generate}}{\rightsquigarrow}$ finite n-fold parallel products

$$x_1 \times x_2 \xrightarrow[\quad]{p_1 \times p_2} y_1 \times y_2 \,,\; 1 \xrightarrow[\quad]{1} 1 \quad \underset{\text{induct on}}{\rightsquigarrow} \quad x_1 \times \cdots \times x_n \xrightarrow[\quad]{p_1 \times \cdots \times p_n} y_1 \times \cdots \times y_n$$

$$\Pi \begin{pmatrix} x_1 \to y_1 \\ x_2 \to y_2 \\ \vdots \\ x_{n-1} \to y_{n-1} \end{pmatrix} \times x_n \to y_n$$

in a double category;

binary & nullary parallel products & braidings ⇝ generate ⇝ finite (id,bij) products

$x_1 \times x_2 \xrightarrow{p_1 \times p_2} y_1 \times y_2$, $1 \xrightarrow{1} 1$

$x \times y \xrightarrow{1_x \otimes 1_y} x \times y$

⇝ $x_1 \times \cdots \times x_n \xrightarrow{\tilde{\prod} p_i} y_{f(1)} \times \cdots \times y_{f(n)}$

thank you :)

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