Quasi-category Theory

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Abstract

A quasi-category is a model for an $(\infty, 1)$-category, that is a category with objects, morphims, morphisms between morphisms, and morphims between morphisms of every higher dimension. Further, every morphism of dimension 2 or higher is invertable. In this report we will discuss how some results from regular category theory can translate into quasi-category theory, in particular we will discuss joins and slices, initial and terminal objects and (co)limits. An introductory exposition of quasi-categories can be found in Markus Land [2] and an in depth exploration can be found in Jacob Lurie [3], good expositions can also be found in Riehl [4] and Kapulkin [1].

Contents

1 Simplicial sets and quasi-categories

In this section we define the framework of quasi-categories and why this is a good model for $(\infty, 1)$-categories. First we define the simplex category and simplicial sets.

Definition 1.1. The *simplex category* Δ is the category whose:

- Objects are posets $[n] = \{0 \leq 1 \leq \cdots \leq n\}$.
- Morphisms are monotone maps.

The composition and identities are canonical.

We call a presheaf $X \colon \Delta^{op} \to \mathbf{Set}$ a *simplicial set* and call the category of presheaves $\mathbf{sSet} = \mathbf{Set}^{\Delta^{op}}$.

Definition 1.2. The following simplicial sets will be significant:

- We call the presheaf representing $[n]$,
$$\Delta^n = sSet(-, [n])$$
the *n-simplex*.
- We call
$$\Lambda^n_k = \bigcup_{k \in E \subsetneq [n]} \Delta^E \subset \Delta^n$$
the *kth horn of* Δ^n.

Given a simplicial set X we write $X([n]) = X_n$ for its set of n-simplicies and by the Yoneda Lemma we have each n-simplex $x \in X_n$ corresponds to a map $x \colon \Delta_n \to X$. We will call the set of 0-simplicies the *objects* of X and X_1 the *morphisms* of X.

Definition 1.3. The *n-skeleton* is the functor $sk_n \colon \mathbf{sSet} \to \mathbf{sSet}$ defined as $sk_n(X) = i_! i^*(X)$ where $i \colon \Delta_{\leq n} \hookrightarrow \Delta$ is the inclusion of the full subcategory of Δ with objects $[m]$ for $m \leq n$. The *n-coskeleton* is the functor $cosk_n \colon \mathbf{sSet} \to \mathbf{sSet}$ defined as $cosk_n(X) = i_* i^*(X)$. Concreteley, we have the k-simplicies,
$$cosk_n(X)_k = \mathbf{sSet}(sk_n(\Delta^k), X).$$

Remark 1.4. $sk_n \dashv cosk_n$.

Definition 1.5. There is a fully faithful embedding of $\mathbf{Cat}$ in to $\mathbf{sSet}$ called the Nerve, denoted $N \colon \mathbf{Cat} \to \mathbf{sSet}$. This functor admits a left adjoint called the realisation $\tau \colon \mathbf{sSet} \to \mathbf{Cat}$.

We omit the proof of the following Lemma, which can be found in [5].

Lemma 1.6. The nerve of a category is 2-coskeletal, that is $N\mathscr{C} \cong cosk_2 N\mathscr{C}$.

1.1 Quasi-categories

Here we will build up some fundamental ideas of quasi-categories and how they are a natural extension of categories.

Definition 1.7. A *quasi-category* is a simplicial set $C\colon \Delta^{op} \to \mathbf{Set}$ such that for any integers $n \geq 2$ and $0 < k < n$ any morphism of the form $\Lambda^n_k \to C$ extends to a morphism $\Delta^n \to C$. A map of quasi-categories is a map of simplicial sets and therefore the category of quasicategories is a full subcategory of **sSet**

We call the horns Λ^n_k for $0 < k < n$, inner horns. Recall that if we included the endpoints this would be the definition of a Kan complex, and so every Kan complex is a quasi-category. We will now see that the nerve functor also provides an example of a quasi-category.

Definition 1.8. The category **Cat** of locally small category fully embeds into the category **sSet** via the nerve functor $N\colon \mathbf{Cat} \to \mathbf{sSet}$ defined as $N(\mathscr{C})_n = \mathbf{Cat}([n], \mathscr{C})$.

Remark 1.9. The nerve of $[n]$ is $N([n]) = \Delta_n$.

The following proposition and proof are thanks to Riehl and Verity [5].

Proposition 1.10. Given a category $\mathscr{C}$ the nerve, $N\mathscr{C}$ is a quasi-category.

Proof. We have that the nerve of a category is isomorphic to its 2-skeleton by 1.6 so $N\mathscr{C} \cong \mathscr{C}$, via the adjuntion in 1.4 we have the lifting problem,

$$\begin{tikzcd} sk_2\Lambda^n_k \arrow[r] \arrow[d] & N\mathscr{C} \\ sk_2\Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

For $n \geq 4$ the inclusion $sk_2\Lambda^n_k \hookrightarrow sk_2\Delta^n$ is an isomorphism and hence the lift has *unique* solutions. The remaining horn for $n = 2$ has a lift since two morphims in a category admit a *unique* composite and the remaining horns for $n = 3$ admit a lift since composition is associative up to equality and hence this is also a *unique* filler. □

The following corollary can be found in Land [2] (Thm 1.1.52).

Corollary 1.11. Let C be a quasi-category. If all inner horn inclusions $\Lambda^n_k \to \Delta^n$ for $0 < k < n$, $n \geq 2$ admit a unique filler then there are unique fillers $sp_n \to \Delta^n$ for $n \geq 2$ and $C \cong N\mathscr{C}$ for some category $\mathscr{C}$. The associated diagram is:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & C \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proof. The full proof can be found in Land [2]. Suppose $C \cong N\mathscr{C}$ for some category $\mathscr{C}$ then if $n = 2$ we have one inner horn, Λ^2_1. Suppose the filler is not unique, since $C \cong N\mathscr{C}$ we have a pair of composable morphisms in $\mathscr{C}$, f, g such that when we fill the horn in $N\mathscr{C}$ we get two different composites,

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\alpha'"] & \\ x \arrow[ur, "f"] \arrow[rr, "h'"'] & {} & z \end{tikzcd}$$

Then we must have $h = h'$ and $\alpha = \alpha'$ since, in $\mathscr{C}$ the composition of two morphisms is unique. For $n \geq 3$ we can proceed by induction on n (note that $\Delta^{n+1} \cong \Delta^n \star \Delta^0$ see 1.19 below).

Suppose C is a quasi-category with unique horn liftings. Then we can construct a category $\mathscr{C}$ as follows: The objects are the 0-simplicies of C. The morphims from x to y are 1-simplicies of C such that for a 1-simplex f $\partial_0 f = y$ and $\partial_1 f = x$. The identity at x is the degeneracy $\sigma_0 x$. Composition is given by the unique horn filling $\Lambda^2_1 \to \Delta^1$. It can be checked that this is a well defined category. It remains to show that $C \cong N\mathscr{C}$. Since an n-simplex of C is determined by the restriction to the spine we see a map $p\colon C \to N\mathscr{C}$ has to be isomorphic on 0 and 1-simplicies and also satisfy the following commutative square.

$$\begin{tikzcd} C_n \arrow[r] \arrow[d] & N\mathscr{C}_n \arrow[d] \\ C_1 \times \cdots \times C_1 \arrow[r] & N\mathscr{C}_1 \times \cdots \times N\mathscr{C}_n \end{tikzcd}$$

We have that the left and right maps are isomorphism and the bottom map is an isomorphism since C and $N\mathscr{C}$ are isomorphic on 1-simplicies. Therefore, by $2 - of - 3$ the top map is an isomorphism. $\square$

Here we will state informally that the theory of categories extends to the theory of quasi-categories, that is the concepts in quasi-categories become there related concepts in categories when restricted along the nerve functor.

We can define a similar notion of a functor quasi-category as follows,

Definition 1.12. Let X be a simplicial set and C a be quasi-category. Define the quasi-category of functors between X and C as $Fun(X, C) = C^X$.

We proved in lectures for a simplicial set C^X is a quasi-category when C is a quasi-category. The following is the quasi-category definition for opposite categories, can be found in [3].

Definition 1.13. Let X be a simplicial set. We define X^{op} as the simplicial set with n-simplicies, $X^{op}_n = X_n$, face maps;

$$d_i\colon X^{op}_n \to X^{op}_{n-1} = d_{n-i}\colon X_n \to X_{n-1}$$

and degeneracy maps,

$$\sigma_i\colon X^{op}_n \to X^{op}_{n+1} = \sigma_{n-1}\colon X_n \to X_{n-1}.$$

Remark 1.14. If C is a quasi-category then C^{op} is a quasi-category.

Definition 1.15. We call a simplicial set X *contractible* if there exits a simplicial homotopy $H\colon X \times \Delta^1 \to X$ such that $H|_{\partial\Delta^1} = [const_x, id_X]$ where $const_x$ is the constant map at an object $x \in X$ and id_X is the identity map on X.

The following criterion for contractibility is thanks to Kapulkin [1] (3.3.2)

Proposition 1.16. Let X be a simplicial set, then X is contractible if and only if for all $n \geq 2$ and maps $\partial\Delta^n \to X$ we have an extension to $\Delta^n \to X$ that is,

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

commutes.

Lemma 1.17. Let X be a simplicial set. X is contractible if and only if X^{op} is contractible.

Proof. Here we give one direction, the other is dual. Suppose X is contractible, then we have a homotopy $H\colon X \times \Delta^1 \to X$ with $H|_{\partial\Delta^1} = [const_x, id_X]$. But then we can construct the homotopy $H^{op}\colon X^{op} \times \Delta^1 \to X^{op}$ with $H|_{\partial\Delta^1} = [const_x, id_{X^{op}}]$. □

The following proposition can be found in Land [2] here we give the dual proof.

Proposition 1.18. Let $\mathscr{C}$ be a category. If $\mathscr{C}$ has an initial or terminal object then $N(\mathscr{C})$ is contractible.

Proof. We give the proof for $\mathscr{C}$ with a terminal object, 1. We have functors $Const_1\colon \mathscr{C} \to \mathscr{C}$ and $id_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$. There is a natural transformation $\alpha\colon id_{\mathscr{C}} \Rightarrow Const_1$ where for each object $X \in \mathscr{C}$, $\alpha_X = X \to 1$. Naturality,

$$\begin{tikzcd} X \arrow[r, "\alpha_x"] \arrow[d, "f"'] & 1 \arrow[d, "id_1"] \\ Y \arrow[r, "\alpha_y"'] & 1 \end{tikzcd}$$

follows from uniqueness of $X \to 1$. The natural transformation can be seen as a functor $H_\alpha\colon X \times [1] \to X$ with $H|_{[1]} = [id_{\mathscr{C}}, const_1]$ and then passing through the nerve functor gives a simplicial homotopy, $NH_\alpha\colon N\mathscr{C} \times \Delta^1 \to N\mathscr{C}$ with $NH_\alpha|_{\Delta_1} = [id_{N\mathscr{C}}, const_1]$. This can be viewed as a homotopy $H^{op}_\alpha\colon N\mathscr{C}^{op} \times \Delta^1 \to N\mathscr{C}^{op}$ with $H^{op}_\alpha|_{\partial\Delta^1} = [const_{1^{op}}, id_{X^{op}}]$ so $N\mathscr{C}^{op}$ is contractible and therefore $N\mathscr{C}$ is contractible by lemma 1.17. □

Definition 1.19. Let X and Y be simplicial sets. We define the join $X \star Y$ as

$$(X \star Y)_n = \bigsqcup_{i+j=n-1} X_i \times Y_j.$$

Definition 1.20. For a simplicial set X we define the functor $X \downarrow -$ as the right adjoint to $X \star -$ and $- \downarrow X$ as the right adjoint to $- \star X$.

Remark 1.21. Notice that $\star$ is non symmetric, so $X \star -$ is not isomorphic to $- \star X$ in general.

We have a notion of composition of morphims in quasi-categories which we will now discuss. The following is from [4].

Remark 1.22. We can characterise the Λ^2_1 horn as the pushout,

$$\begin{tikzcd} \Delta^0 \arrow[r, "d_0"] \arrow[d, "d_1"'] & \Delta^1 \arrow[d] \\ \Delta^1 \arrow[r] & \Lambda^2_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd},$$

notice that all the morphisms are cofibrations.

Recall that a morphism in a quasi-category C is a one simplex of C, or in other words a map $f\colon \Delta^1 \to C$. So we can hom in to a quasi-category C with the pushout above to obtain the pullback,

$$\begin{tikzcd} C^{\Delta^0} \arrow[dr, phantom, "\lrcorner", very near start] & C^{\Delta^1} \arrow[l, "d_0^*"'] \\ C^{\Delta^1} \arrow[u, "d_1^*"] & C^{\Lambda^2_1} \arrow[l] \arrow[u] \end{tikzcd}$$

We have that all the morphisms in this pullback are now fibrations. We can think of $C^{\Lambda^2_1}$ as the quasi-category of composable arrows in C. Also notice $C^{\Delta^1} \cong C$ so the morphisms d_ε^* are projections to C which can be thought of as source and target maps.

2 Initial and terminal objects

In this section we will see the notion of initial and terminal objects in terms of quasi-categories. This will later be used to define (co)limits in quasi-categories.

Definition 2.1. Let C be a quasi-category and x, y objects of C. we define the mapping space of x and y as the pullback,

$$\begin{tikzcd} map_C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\Delta^1} \arrow[d] \\ \Delta^0 \arrow[r, "{(x,y)}"] & C \times C \end{tikzcd}.$$

The following definition can be found in Land [2]

Definition 2.2. Let C be a quasi-category. An object i of C is said to be *initial* if for all objects y of C the mapping space $map_C(i,y)$ is contractible. An object t of C is said to be *terminal* if for all objects y of C the mapping space $map_C(y,t)$ is contractible.

From here on we will just look at the case for terminal objects as all the ideas for initial objects follow dually. We have the following alternative characterizations of and terminal objects thanks to [2].

Lemma 2.3. Let C be a quasi-category and t and object of C then the following are equivalent:

1. t is terminal.
2. $C \downarrow t \to C$ is a trivial fibration.
3. For $1 \leq n$ the every lifting problem:

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "t"] & \partial\Delta^n \arrow[r] \arrow[d] & C \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

admits a solution.

Proof. We include the proof for the equivalence of 2 and 3. The complete proof can be found in Land [2]. The following map

$$\partial\Delta^n \star \Delta^0 \sqcup_{\partial\Delta^{n-1}\star\varnothing} \Delta^{n-1} \star \varnothing \to \Delta^{n-1} \star \Delta^0$$

is equivalent to the map $\partial\Delta^n \hookrightarrow \Delta^n$. Therefore the lifting problem,

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "t"] & \partial\Delta^n \arrow[r] \arrow[d] & C \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

is equivalent to the lifting problem,

$$\begin{tikzcd} \partial\Delta^{n-1} \arrow[r] \arrow[d] & C \downarrow t \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & C \end{tikzcd} .$$

Where we get a lift by part 2 of the claim. □

We get the following corollary which gives us a familiar result from category theory that terminal objects are unique up to isomorphism.

Corollary 2.4. Let C be a quasi-category then the full subcategory of terminal objects, C_{term} is either empty or a contractible Kan complex.

Proof. Suppose C_{term} is non empty. For all n We need to solve the following lifting problem,

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & C_{term} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} .$$

For $n = 0$ there is a solution since C_{term} is non-empty. For $n \geq 1$ we have a lift by Lemma 2.3. □

We can now prove a familiar result for initial and terminal objects. The following proposition can be found in Land [2] (Proposition 4.1.5).

Proposition 2.5. Let X be a simplicial set and C a quasi-category and $F\colon X \to C$ a functor. If for all objects $x \in X$, $F(x)$ is terminal then F is terminal in $Fun(X, C)$.

Proof. Here is the proof for terminal objects. We aim to solve the lifting problem.

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "F"] & \partial\Delta^n \arrow[r, "\hat{F}"] \arrow[d] & Fun(X, C) \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

For $n \geq 1$. By adjointness this reduces to solving,

$$\begin{tikzcd}
X \times \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "F"] & X \times \partial\Delta^n \arrow[r] \arrow[d] & C \\
& X \times \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

Where F maps objects $x \in X$ to terminal objects in C. We can consider the filtration, $F_k(X) = sk_k(X \times \Delta^n) \cup X \times \partial\Delta^n$ and so we need to solve the following lifting problem,

$$\begin{tikzcd}
\coprod \partial\Delta^k \arrow[r] \arrow[d] & F_{k-1}(X) \arrow[r] \arrow[d] & C \\
\coprod \Delta^k \arrow[r] & F_k(X) \arrow[ur, dashed] &
\end{tikzcd}.$$

We proceed inductively and see the composite of the top map sends a vertex to a terminal object in C and so the diagram admits a lift for k since we have the map $F_{k-1}(X) \to C$. Therefore passing to the colimit over k we get the lift for the original problem. □

Corollary 2.6. Let C be a quasi-category with a terminal object t. Then for a simplicial set X any terminal object in $Fun(X, C)$ takes values in C_{term}.

Proof. Let c_t be the constant functor at t. By 2.5 c_t is terminal in $Fun(X, C)$. Every terminal object $F \in Fun(X, C)_{term}$ is weakly equivalent to c_t since $Fun(X, C)$ is a contractible Kan complex. So for every object $x \in X$, $c_t(x)$ is weakly equivalent to $c_t(x)$ hence a terminal object in C. □

3 Limits and colimits

In this section we will see how limits and colimits in regular category theory can be translated to quasi-categories. As in regular category theory there are different ways to talk about (co)limits, here we will introduce them via universal cones over diagrams as in Land [2], we will then relate it to the definition given in Lurie [3] as terminal/initial objects of some slice category.

3.1 (Co)limits via universal cones

Definition 3.1. Let $F\colon X \to C$ be a functor and x be an object of C. W define the simplical set $map_C(F, x)$ as the pullback,

$$\begin{tikzcd}
map_C(F,x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Fun(X \star \Delta^0, C) \arrow[d] \\
\Delta^0 \arrow[r, "{(F,x)}"'] & Fun(X, C) \times C
\end{tikzcd}$$

Where right map is given by,

$$\begin{tikzcd}
Fun(X \cup \{\infty\}, C) \arrow[r, hook] \arrow[d] & Fun(X \star \Delta^0, C) \arrow[dl] \\
Fun(X, C) \times C &
\end{tikzcd}.$$

Remark 3.2. If F is the functor $F = y\colon \Delta^0 \to C$, that is F is an object in C then $map_C(F, x) = map_C(y, x)$, the mapping space between the objects x, y.

Here we should view the functor F as a diagram of C which we wish to take the limit over. We have the following proposition from Land [2] (Proposition 4.3.2),

Proposition 3.3. $map_C(F, x)$ defined above is a Kan complex.

Proof. We have that the right map is an inner fibration since its a fibration between quasi-categories. Furthermore, this map is conservative. Therefore, the map on the right is a conservative inner fibration since fibrations are stable under pullbacks. Therefore, $map_C(F, x)$ is a Kan complex. □

Definition 3.4. Dual to Definition 3.1 we have the simplicial set defined by the pullback,

$$\begin{tikzcd} map_C(x, F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Fun(\Delta^0 \star X, C) \arrow[d] \\ \Delta^0 \arrow[r, "{(x,F)}"'] & C \times Fun(X, C) \end{tikzcd} .$$

We now see the quasi-category view of universal cones over a diagram $F\colon X \to C$. The following definition is thanks to Kapulkin [1].

Definition 3.5. Let C be a quasi-category and $F\colon X \to C$ a functor (a diagram). A *cone under F* is a simplicial map $\bar{F}\colon X \star \Delta^0 \to C$ such that the following diagram,

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, hook] & C \\ X \star \Delta^0 \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$

commutes. With abuse of notation we define a *cone over F* as a simplicial map $\bar{F}\colon \Delta^0 \star X \to C$ such that,

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, hook] & C \\ \Delta^0 \star X \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$

commutes.

Definition 3.6. Let $F\colon X \to C$ be a functor (a diagram) and $\bar{F}_c\colon X \star \Delta^0 \to C$ a cone over F. $\bar{F}_c$ is *universal* or a *colimit* if for all objects x of C the map,

$$map_C(\bar{F}_c, x) \to map_C(F, x)$$

is a homotopy equivalence. Let $\bar{F}_l\colon$ be a cone under F. $\bar{F}_l$ is *universal* or a *limit* if the map,

$$map_C(x, \bar{F}_l) \to map_C(x, F)$$

is a homotopy equivalence.

We have an alternative definition of a universal cone over a diagram found in Kapulkin [1] which follows from Proposition 1.16.

Lemma 3.7. Let $\bar{F}_c\colon X \star \Delta^0 \to C$ be a cone over F, we say $\bar{F}_c$ is a colimit if for all $n > 0$ and $H\colon X \star \partial\Delta^n \to C$ with $H|_X = \bar{F}_c$, we have there exists a map $X \star \Delta^n \to C$ such that,

$$\begin{tikzcd} X \star \partial\Delta^n \arrow[r] \arrow[d, hook] & C \\ X \star \Delta^n \arrow[ur, dashed] & \end{tikzcd}.$$

commutes. Let $\bar{F}_l\colon \Delta^0 \star X \to C$ be a cone under F, we say $\bar{F}_l$ is a limit if for all $H\colon \partial\Delta^n \star X \to C$ we have there exists a map $\Delta^n \star X \to C$ such that,

$$\begin{tikzcd} \partial\Delta^n \star X \arrow[r] \arrow[d, hook] & C \\ \Delta^n \star X \arrow[ur, dashed] & \end{tikzcd}.$$

commutes.

Example 3.8. Let C be a quasi-category and $F\colon \Lambda^2_0 \to C$. The colimit of F is called a pushout. In this case we have $Fun(\Lambda^2_0 \star \Delta^0, C) \cong Fun(\Delta^1 \times \Delta^1, C)$

Remark 3.9. The limit of $F\colon \Lambda^2_2 \to C$ is a pullback.

Example 3.10. Let C be a quasi-category and D be a discrete set viewed as a category and $F\colon D \to C$ be a functor. The colimit of F is called a coproduct.

Example 3.11. Let C be a quasi-category the limit of the functor $\varnothing \to C$ is a terminal object in C. This agrees with C_{term} being contractible and 3.13 below.

The following theorem is thanks to Land [2].

Theorem 3.12. Let C be a quasi-category and $F\colon X \to C$ be a functor (a diagram). The cone $\bar{F}\colon X \star \Delta^0 \to C$ is a colimit if and only if it is an initial object in $F \downarrow C$.

Proof. We have the commutative diagram,

$$\begin{tikzcd} \bar{F} \downarrow C \arrow[r] \arrow[d] & F \downarrow C \arrow[dl] \\ C & \end{tikzcd}.$$

From an assignment we have the isomorphism $\bar{F} \downarrow C \cong y \downarrow F \downarrow C$, so the horizontal map is an equivalence if and only if $\bar{F}$ is an initial object. Then by Theorem 3.1.27 and Lemma 4.3.9 in Land [2] we see that this map is an equivalence if and only if the map,

$$map_C(\bar{F}, x) \to map_C(F, x),$$

which is exactly when $\bar{F}$ is a colimit. $\square$

We recover the result from category that colimits are unique up to a unique isomorphism.

Theorem 3.13. Let C be a quasi-category and $F\colon X \to C$ be a functor (a diagram). Then the full subcategory of colimits $F \downarrow C$ is either empty or a contractible Kan complex.

Proof. This proof is similar as for initial and terminal objects. □

This reflects the case for regular category theory since when we pass to the homotopy category of a quasi-category the subcategory of colimits being a Kan complex means there exist isomorphisms between the all the objects of the homotopy category of that subcategory, It being contractible means these choices are unique.

Remark 3.14. There is also a definition of colimits in terms of representable functors $F\colon C \to Spc$ where Spc is the category of spaces. We will not talk about it here but details can be found in Land [2].

Remark 3.15. All the results stated above can be dualised to limits.

Definition 3.16. We call a simplicial set *small* it is weakly equivalent to a simplicial set where collection of non-degenerate simplicies can form a set. We say a quasi-category has all small colimits if for all finite simplicial sets X and functors $F\colon X \to C$ there exists a universal cone.

We include the following result without proof but that can be found in Land [2] (Proposition 4.3.28)

Proposition 3.17. If a quasi-category C has all coproducts and pushouts then it has all small colimits.

3.2 Adjunctions

We can characterise adjunctions using slices and initial object which we present here, however, there is a standard definition which we will not include here since we haven't spoke about (co)cartesian fibrations. You can find the definition in Kapulkin [1].

The following is

Definition 3.18. Let A, B, C be quasi-categories and $f\colon B \to A$ and $g\colon C \to A$ be functors. We define the slice $f \downarrow g$ as the pullback,

$$\begin{tikzcd} f \downarrow g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Fun(\Delta^1, A) \arrow[d] \\ B \times C \arrow[r, "f\times g"'] & A \times A \end{tikzcd}$$

Remark 3.19. $f \downarrow g$ defined above is a quasi-category and there are projections $e_0\colon f \downarrow g \to C$ and $e_1\colon f \downarrow g \to B$.

Definition 3.20. Let C and D be quasi-categories and $F\colon C \to D$, $G\colon D \to C$ then $F \vdash G$ is an *adjunciton of quasi-categories* if and only if there is an equivalence,

$$F \downarrow D \cong C \downarrow G,$$

over $D \times C$.

Proposition 3.21. Let C and D be quasi-categories and $G\colon C \to D$ be a simplicial map. G has a *left adjoint* if and only if for each $x \in C$ the quasi-category $x \downarrow G$ has an initial object.

References

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